MRI: Assignment #1

Due Tuesday, Nov 2nd, 2021

Rotations

Write down the expression for a left handed rotation matrix about the y-axis by 90°. Apply it to a unit vector along the z-axis and verify your result makes sense (i.e. what would you get by rotating your vector about the y-axis by 90° using the left hand rule? Is that the actual vector you got?)

Rotations Do Not Commute

You're probably used to mathematical operations that commute, meaning it doesn't matter in which order you apply them. For example, 4×5=5×4, or 11+12=12+11. However, this is not the case for rotations or matrix multiplication: Show that a rotation about z by 90° followed by a rotation about y by 90° is not the same as a rotation about y by 90° followed by a rotation about z by 90°.

Complex Numbers

1. Use complex numbers and Euler's identity ($e^{ix}=\cos(x)+i\sin(x)$) to prove that $\sin^2(x)+\cos^2(x)=1$ (Hint: explain why $1=e^{ix}e^{-ix}$, and use it).
2. Consider the complex number $v(t)=v_0e^{i\omega t}$. Prove that its magnitude remains constant. Treat it as a 2D vector in the (real, imaginary) plane, and plot it for different values of *t*. Convince yourself its tip executes circular motion. How long will it take it to complete a circle? (i.e. what is its period?)

SNR & Averaging

You are given two random numbers, n_1, n_2, taken from the same random distribution with zero mean and some fixed standard deviation[1] (SD):

$$\text{mean} \equiv \langle n_1 \rangle = \langle n_2 \rangle = 0$$
$$SD \equiv \sqrt{\langle n_1^2 \rangle} = \sqrt{\langle n_2^2 \rangle} = \sigma$$

They are uncorrelated, so their product is zero (on average - i.e. if we were to draw pairs of numbers repeatedly):

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[1] $\langle x \rangle$ stands for the mean of x. It is the number you would get if you would draw *x* over and over and calculate the average. It is linear, so if a and b are numbers and x_1, x_2 are random variables, then $\langle ax_1 + bx_2 \rangle = a\langle x_1 \rangle + b\langle x_2 \rangle$. The standard deviation is the mean of the deviation from the mean: $SD = \sqrt{\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle}$.

$$\langle n_1 n_2 \rangle = 0$$

Prove that the SD of the sum $n_1 + n_2$ is equal to $\sqrt{2}\sigma$. What would be the SD if we took M random numbers and added them up? Use your results to argue that, if we acquire the same image twice (each image will contain random noise) and add up the images we would get an increase in SNR of only $\sqrt{2}$, and not 2 as one might naively assume. This means that **the SNR in MRI grows as the square root of the acquisition time** (that is, if in T minutes you could get an image with an SNR of X, then, given αT minutes you could only improve the SNR by a factor of $\sqrt{\alpha}$).

Convolution and The Point Spread Function (PSF)

The spatial PSF of a particular sensor is a Gaussian (normalized to unity area, although that is not really necessary):

$$PSF(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

What is the width of this PSF, as a function of σ?
You are now given an "object" you would like to image, made out of two "peaks" at $\pm x_0$:

$$f(x) = e^{-\frac{(x-x_0)^2}{2\beta^2}} + e^{-\frac{(x+x_0)^2}{2\beta^2}} .$$

Calculate the resulting image by convolving both functions. Plot the PSF, objects and resulting image for $\sigma = 1, x_0 = 1, \beta = 0.1$ (i.e. the PSF is much wider than the object's features), and for $\sigma = 0.01, x_0 = 1, \beta = 0.1$ (i.e. the PSF is much narrower than the object's features). Which of the two situations is "desirable"? Explain your choice.

Hint: use the following identity as needed (valid for α<0):

$$\int_{-\infty}^{\infty} e^{\alpha x^2 + \beta x + \gamma} dx = \sqrt{\frac{\pi}{-\alpha}} e^{-\frac{\beta^2}{4\alpha} + \gamma}$$