MRI Primer: Assignment #3 Solution

Thermal Equilibrium Signal

1. A first approximation would assume the voxel is made completely out of water, which has a molecular weight of about 18 gr/mole. A 1 mm^3 voxel of water would weigh about 10^{-3} grams and therefore contains $N_w = \frac{10^{-3}}{18} \text{Moles} \approx 5.5\times10^{18}$ water molecules, and twice the number of protons (remember, it's all about the number of protons, not molecules!), so $N\approx10^{19}$. More careful estimates might take into account that most soft tissues in our body have a density close to 1 gr/mL, or 10^{-3} gr/mm^3. About 65% of that is water, implying we need to simply scale N by 65%, $N\approx0.65\times10^{19}$
2. This is a simple application of the equation for the bulk magnetization:

$$M_0^{(\text{bulk})} = \frac{N(\gamma\hbar)^2 B_0}{4kT}.$$

Here

- B_0 is 3 Tesla
- k Boltzmann's constant, 1.38×10^{-23} Joule/Kelvin
- T Room temperature, 293 Kelvin (25° C)
- $\hbar$ Planck's constant, 1.05×10^{-34} Joule·sec
- γ Proton gyromagnetic ratio, $2\pi\times42.576\times10^6$ Hz/Tesla
- N Number of protons in the voxel

Using these numbers:

$$M_0^{(\text{bulk})} \approx 10^{-12}\frac{J}{T}$$

Signal Induction

1. We will use the expressions derived in class:

$$M_z(t) = M_z(t=0)e^{-t/T_1} + \left(1-e^{-t/T_1}\right)M_0$$
$$M_{xy}(t) = M_{xy}(0)e^{-i\gamma B_0 t}e^{-t/T_2}$$

The real and imaginary parts of $M_{xy}(t)$ are M_x and M_y. We know that at time t=0 $\mathbf{M}(0) = (M_{x0}, 0, 0)$, which also means that $M_{xy}(0) = M_x(0) + iM_y(0) = M_{x0}$, so:

$$M_x(t) = \text{Re}\left[M_{xy}(t)\right] = M_{x0}e^{-t/T_2}\cos(\gamma B_0 t)$$
$$M_y(t) = \text{Im}\left[M_{xy}(t)\right] = -M_{x0}e^{-t/T_2}\sin(\gamma B_0 t)$$

2. The principle of reciprocity states that the signal in the coil will be given by

$$v = -\mathbf{B} \cdot \frac{d\mathbf{M}}{dt}$$

where **B** is the field created at the position of the magnetic moment by putting a unit current through the loop. For this loop,

$$\mathbf{B} = \frac{\mu_0}{2r}\hat{\mathbf{y}} .$$

Thus:

$$v = -\left(\frac{\mu_0}{2r}\hat{\mathbf{y}}\right) \cdot \left(\frac{dM_x}{dt}\hat{\mathbf{x}} + \frac{dM_y}{dt}\hat{\mathbf{y}}\right) = -\frac{\mu_0}{2r}\frac{dM_y}{dt}$$

since $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$ (the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthogonal) and $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1$. The derivative of M_y is

$$\begin{aligned} \frac{dM_y(t)}{dt} &= -M_{x0}\frac{d\left[e^{-t/T_2}\sin(\gamma B_0 t)\right]}{dt} \\ &= -M_{x0}\left[\frac{d\left(e^{-t/T_2}\right)}{dt}\sin(\gamma B_0 t) + e^{-t/T_2}\frac{d\left(\sin(\gamma B_0 t)\right)}{dt}\right] \\ &= -M_{x0}\gamma B_0\left[-\frac{\sin(\gamma B_0 t)}{\gamma B_0 T_2} + \cos(\gamma B_0 t)\right]e^{-t/T_2} \end{aligned}$$

and

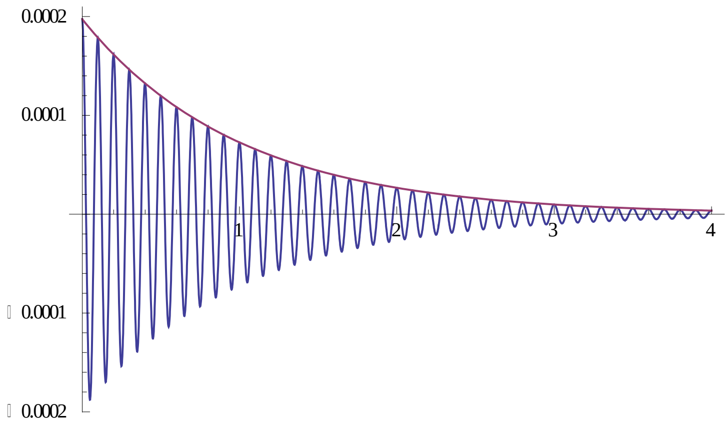
$$v(t) = \frac{\mu_0 M_{x0}\gamma B_0}{2r}\left[-\frac{\sin(\gamma B_0 t)}{\gamma B_0 T_2} + \cos(\gamma B_0 t)\right]e^{-t/T_2} .$$

Now, in reality, γB_0 is about $2\pi \cdot 123$ MHz at 3 Tesla, while T_2 is on the order of 10-100 ms, meaning $\gamma B_0 T_2 \approx 10^7$ or larger, making the first term $\frac{\sin(\gamma B_0 t)}{\gamma B_0 T_2}$ completely negligible compared to the first, so

$$v(t) \approx \frac{\mu_0 M_{x0}\gamma B_0}{2r}\cos(\gamma B_0 t)e^{-t/T_2} .$$

For the next step, however, we'll be keeping it because I chose ω and T_2 that are not that far apart.

3. Taking r=0.2 meters, $\mu_0 = 4\pi \cdot 10^{-7}\ N \cdot A^{-2}$, T_2=1 sec, $\omega=\gamma B_0=10\cdot 2\pi$ Hz·rad, M_{x0}=1 J/T, we obtain for t between 0 and 4 seconds:



The voltage v(t) (blue line) decays with a time constant T_2 and oscillates with a time constant $\omega = \gamma B_0$. The purple line is simply the exponential envelope:

$$\frac{\mu_0 M_{x0} \gamma B_0}{2r} e^{-t/T_2}$$

Note that the values of v(t) (on the y-axis) are not very meaningful since some of the parameters we chose were arbitrary/non-realistic (small ω, M_{x0}=1).

4. If we neglect T_1 relaxation (which is "slow", making this a good approximation), the signal will be 0. This can be seen immediately from the principle of reciprocity: the magnetization vector (and its derivative) lies in the xy plane, while the normal to the loop's surface points along z, and therefore their dot product is zero. In terms of magnetic flux, if we draw the magnetic field lines of **M** in the xy-plane we see that they are all pointing in the xy-plane itself and have no component perpendicular to the coil loop. Therefore no magnetic flux passes through the coil. Now, it is true that the **derivative** of the magnetic flux is what induces a voltage, but to have a non-zero derivative we must have non-zero flux to begin with!
If we take T_1 relaxation into account, then (using $M_z(t=0)=0$, i.e. initially there is no magnetization along the z-axis):

$$M_z(t) = \left(1 - e^{-\frac{t}{T_1}}\right) M_0$$
$$\frac{dM_z}{dt}(t) = \frac{M_0}{T_1} e^{-\frac{t}{T_1}}$$

The magnetic field created by the loop points along the z-axis, and is given by (for unit current):

$$\boldsymbol{B} = B_z\hat{\boldsymbol{z}} = \frac{\mu_0}{2r}\hat{\boldsymbol{z}}$$

Using the principle of reciprocity, the induced voltage is:

$$v = -\mathrm{B_z} \cdot \frac{\mathrm{dM_z}}{\mathrm{dt}} = -\frac{\mu_0 M_0}{T_1 2r} e^{-\frac{t}{T_1}}$$

Feel free to check this induced voltage is MUCH smaller than the voltage induced in part (3) of this question. The reason is that M_z changes very slowly (on the order of $1/T_1$) compared to M_x and M_y (which oscillate with a frequency $\omega_0 = \gamma B_0 \gg 1/T_1$).

Frame Transformations

1. There are several ways to approach this. The easiest one is to realize that $\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}'$ perform a circular motion in the xy plane, starting (respectively) from the x and y axes at time t=0. This means that, at time t=0, and denoting components in the xyz frame

$$\hat{\boldsymbol{x}}'(t=0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{xyz} \quad , \quad \hat{\boldsymbol{y}}'(t=0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{xyz}$$

Their LH rotation can be described using a LH rotation matrix about the z-axis:

$$R_z(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that, for example,

$$\hat{\boldsymbol{x}}'(t) = R_z(\omega t)\hat{\boldsymbol{x}}'(t=0) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{xyz}$$
$$= \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \\ 0 \end{pmatrix}_{xyz}$$

and similarly

$$\hat{\boldsymbol{y}}'(t) = R_z(\omega t)\hat{\boldsymbol{y}}'(t=0) = \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}_{xyz}$$

This notation is equivalent to saying that the components of $\hat{\boldsymbol{x}}'$ in the (xyz) system are $\cos(\omega t), -\sin(\omega t)$ and 0, which is equivalent to writing

$$\hat{\boldsymbol{x}}'(t) = \underbrace{\cos(\omega t)}_{a_{11}} \hat{\boldsymbol{x}} - \underbrace{sin(\omega t)}_{a_{12}} \hat{\boldsymbol{y}} + \underbrace{0}_{a_{13}} \cdot \hat{\boldsymbol{z}}$$

Similarly,

$$\hat{\boldsymbol{y}}'(t) = \sin(\omega t)\,\hat{\boldsymbol{x}} + \cos(\omega t)\,\hat{\boldsymbol{y}} + 0 \cdot \hat{\boldsymbol{z}}$$

The easiest identity is $\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}'$, because both are collinear and do not vary as a function of time. This means $a_{31} = a_{32} = 0$, $a_{33} = 1$.
The inverse transformation can be derived in an exactly analogous way, by noting that the $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ vectors appear to execute a **right handed** rotation around the z-axis when viewed from the (x'y'z') frame. A right handed rotation matrix around z' is given by

$$R_{z'}(\omega t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}_{x'y'z'}$$

Analogously,

$$\hat{\boldsymbol{x}}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{x'y'z'}, \quad \hat{\boldsymbol{y}}(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{x'y'z'}$$

This is another way for writing, e.g., $\hat{\boldsymbol{x}}(t = 0) = \hat{\boldsymbol{x}}'$. Carrying out the matrix-vector multiplication we obtain

$$\begin{aligned} \hat{\mathbf{x}}(\mathrm{t}) &= \cos(\omega t)\,\hat{\mathbf{x}}'(t) + \sin(\omega t)\,\hat{\mathbf{y}}'(t) + 0 \cdot \hat{\mathbf{z}}'(t) \\ \hat{\mathbf{y}}(\mathrm{t}) &= -\sin(\omega t)\,\hat{\mathbf{x}}'(t) + \cos(\omega t)\,\hat{\mathbf{y}}'(t) + 0 \cdot \hat{\mathbf{z}}'(t) \\ \hat{\mathbf{z}}(\mathrm{t}) &= 0 \cdot \hat{\mathbf{x}}'(t) + 0 \cdot \hat{\mathbf{y}}'(t) + \hat{\mathbf{z}}'(t) \end{aligned}$$

2. Intuitively, it should be clear that $\mathbf{B}_{\mathrm{RF}}$ (which itself rotations with a frequency ω_{RF} according to a left hand rule) will appear to rotate (with the LH rule) with a frequency $\omega_{RF} - \omega$ in a frame which rotates with a frequency ω. When $\omega = \omega_{RF}$, i.e. when the frame rotates with the same frequency as $\mathbf{B}_{\mathrm{RF}}$, it will appear stationary. This can be derived analytically by plugging in our expressions for $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ into our expression for $\mathbf{B}_{\mathrm{RF}}(\mathrm{t})$

$$\begin{aligned} \boldsymbol{B}_{RF}(t) &= B_1 \cos(\omega_{RF} t)\,[\cos(\omega t)\,\hat{\mathbf{x}}'(t) + \sin(\omega t)\,\hat{\mathbf{y}}'(t)] \\ &\qquad - B_1 \sin(\omega_{RF} t)\,[-\sin(\omega t)\,\hat{\mathbf{x}}'(t) + \cos(\omega t)\,\hat{\mathbf{y}}'(t)] \\ &= B_1(\cos(\omega_{RF} t)\cos(\omega t) + \sin(\omega_{RF})\sin(\omega t))\hat{\boldsymbol{x}}'(t) \\ &\qquad + B_1(\cos(\omega_{RF} t)\sin(\omega t) - \sin(\omega_{RF} t)\cos(\omega t))\hat{\boldsymbol{y}}'(t) \end{aligned}$$

$$= \mathrm{B}_1 \cos\big((\omega_{RF} - \omega)t\big)\, \hat{\boldsymbol{x}}'(t) - \mathrm{B}_1 \sin\big((\omega_{RF} - \omega)t\big)\, \hat{\boldsymbol{y}}'(t)$$

The last step uses two trigonometric identities for the difference of angles within a sine or cosine (Wikipedia is your friend on this one).

3. This is the easiest step: Just substitute $\omega = \omega_{RF}$ above and obtain $\boldsymbol{B}_{RF}(t) = B_1 \hat{\boldsymbol{x}}'$.
4. This is also straightforward: $\mathbf{B} = B\hat{\mathbf{z}} = B\hat{\mathbf{z}}'$, because $\hat{\mathbf{z}} = \hat{\mathbf{z}}'$. This is because the unit vectors along the z and z' axes are the same (because the x'y'z' rotates about the z-axis).

Time Derivatives in the Rotating Frame

1.

$$\frac{d\boldsymbol{M}}{dt} = \frac{d(\cos(\omega t))}{dt}\hat{\boldsymbol{x}} - \frac{d(\sin(\omega t))}{dt}\hat{\boldsymbol{y}}$$
$$= -\omega \cdot (\sin(\omega t)\, \hat{\boldsymbol{x}} + \cos(\omega t)\, \hat{\boldsymbol{y}})$$

The components of the derivatives in the (xy) frame are simply the coefficients of the unit vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$. You could also write this informally as

$$\boldsymbol{M} = \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}_{xy}, \quad \frac{d\boldsymbol{M}}{dt} = \begin{pmatrix} -\omega \sin(\omega t) \\ -\omega \cos(\omega t) \end{pmatrix}_{xy}$$

2. We use the results of the previous problem for transforming between frames:

$$\hat{\boldsymbol{x}} = \cos(\omega t)\, \hat{\boldsymbol{x}}' + sin(\omega t)\hat{\boldsymbol{y}}'$$
$$\hat{\boldsymbol{y}} = -\sin(\omega t)\, \hat{\boldsymbol{x}}' + cos(\omega t)\hat{\boldsymbol{y}}'$$

Substituting this and simplifying we obtain

$$\boldsymbol{M}(t) = \hat{\boldsymbol{x}}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{x'y'}, \quad \frac{d\boldsymbol{M}(t)}{dt} = -\omega\hat{\boldsymbol{y}}' = \begin{pmatrix} 0 \\ -\omega \end{pmatrix}_{x'y'}$$

3. By definition, you need to differentiate the **components** of **M** as they appear in the x'y' frame, which are just 1 and 0:

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = \frac{dM_{x,rot}}{dt}\hat{\boldsymbol{x}}' + \frac{dM_{y,rot}}{dt}\hat{\boldsymbol{y}}' = \frac{d(1)}{dt}\hat{\boldsymbol{x}}' + \frac{d(0)}{dt}\hat{\boldsymbol{y}}' = 0$$

1. Differentiate **M**(t) to obtain d**M**/dt. Express its components in the (xy) frame.
2. Express the components of both vectors, **M** and d**M**/dt, in the (x'y'z') frame, where the (x'y'z') frame is the same as the one described in the previous problem (with a left-handed rotation ω around the z-axis). This tells you what **M** and d**M**/dt would look like to an observer in the x'y'z' frame. Hint: use the expressions for the unit vectors in

the xyz frame in terms of the x'y'z' frame. In particular, show **M** and d**M**/dt are both **constant** and **non-zero**.

3. Now, consider the vector **M**(t) as it appears to an observer in the x'y'z' frame. If asked, what would an observer in the x'y'z' frame (who is unaware of the xyz frame) think the time derivative of **M** should be? In other words, what is $\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot}$? (Hint: it's not the same as d**M**/dt in the x'y'z' frame)

On and Off-Resonance Excitation

1. On resonance - which we assume when analyzing the pulse's duration and RF amplitude based on its flip angle - we have

$$\alpha = (\text{flip angle}) = \gamma B_1 T$$

where T is the pulse's duration. Here $\alpha = \pi/2$, T=1 ms and $\gamma = 2\pi \cdot 42.576\ \frac{\text{kHz}}{\text{mT}}$ for protons, yielding

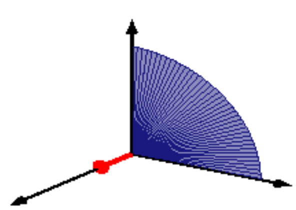
$$B_1 = \frac{\alpha}{\gamma T} \approx 5.9\ \mu T\ .$$

2. The bandwith - that is, the range of offset frequencies excited by the pulse - will be $\bar{\gamma} B_1 = 250\ Hz$.

3. The effective field is simply the RF field, since the offset is zero:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5.9\ \mu T \\ 0 \\ 0 \end{pmatrix}$$

Hence (straight out of the lecture notes!):



The effective field is shown in red, while M is drawn at successive time as a blue vector (essentially starting out from the z-axis and rotating by 90° until it reaches the y-axis).

4. The precession frequency around the effective field, assuming the spin is on resonance, is simply $\nu_{eff} = \bar{\gamma}\left|\mathbf{B}_{eff}\right| = \bar{\gamma} B_1 = 0.25\ \text{kHz}$, the same as the bandwidth.

5. The effective field in the rotating frame:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} = \begin{pmatrix} 5.9\ \mu T \\ 0 \\ 1\ \mu T \end{pmatrix}.$$

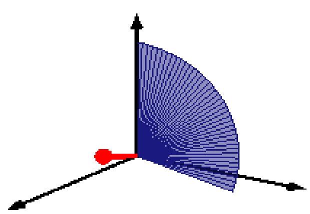
The precession frequency around this field is slightly higher than that around just the RF field without the offset:

$$\nu_{eff} = \not{\gamma} B_{eff} = \gamma\sqrt{B_1^2 + \Delta B^2} \approx 0.254\ \text{kHz}.$$

6. The off-resonance spin precesses about the effective field for the pulse's duration, which is T=1 ms. The total angle by which it precesses around the effective field (note this is **not** along the x-axis!) is:

$$\alpha = \gamma B_{eff} T \approx 1.6\ rad \approx 92°.$$

7. Drawing:



Now B_{eff} has a small z-component which makes it "stick up" in the x-z plane in the rotating frame. Consequently, M precesses not about the x-axis but about this slightly tilted axis in the x-z plane. It also precesses around it by more than 90° (92° to be exact, as we've calculated in the previous part).

Flip Angles Are Nucleus-Dependent

The answer is No. The flip angle depends on the gyromagnetic ratio:

$$\alpha = \gamma B_1 T\ .$$

The flip angle will change by the ratio of gyromagnetic ratios between hydrogen ($\not{\gamma}_H = 42.576\ \frac{kHz}{mT}$) and carbons ($\not{\gamma}_C = 10.705\ \frac{kHz}{mT}$); that is, since

$$\alpha_C = \gamma_C B_1 T$$
$$\alpha_H = \gamma_H B_1 T$$

then (dividing)

$$\alpha_C = \alpha_H \cdot \frac{\gamma_C}{\gamma_H} = 90° \cdot \left(\frac{10.705}{42.576}\right) \approx 23° \ .$$

To achieve α_C of 90° we need to either increase B_1 or T. Both options are in theory valid. In practice, the RF amplitude B_1 is limited by our RF amplifiers and what one ends up doing is making the pulse longer (this creates other issues, e.g. it can make some pulses too long, with T_2 and T_1 starting to have a detrimental effect).