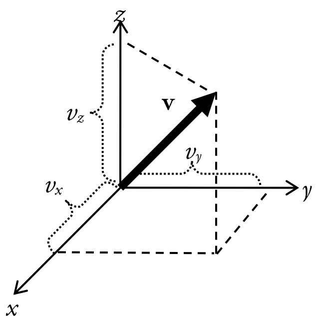
Mathematical Preliminaries

Lecture Notes by Assaf Tal

Vectors And Matrices

A Vector In 3D Is Specified By Three Components

A vector is a geometrical entity that can be thought of as an “arrow” originating from the origins. It can be specified in terms of its projections on the x, y and z axes:



$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Vectors can also be two dimensional in, say, the xy plane:

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Vectors can be added and multplied by a scalar (a number) in the trivial way:

$$a\mathbf{v} + b\mathbf{u} = a\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} + b\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} av_x + bu_x \\ av_y + bu_y \\ av_z + bu_z \end{pmatrix}.$$

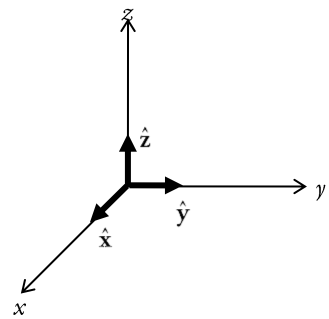
Vector Magnitudes & Unit Vectors

The size (or **magnitude**) of a vector in terms of its components is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}\ .$$

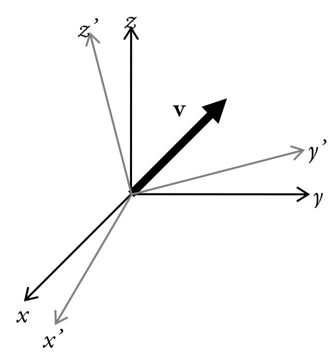
A **unit vector** is a vector with unit size: $|\mathbf{v}| = 1$. Unit vectors are denoted with a hat: $\hat{\mathbf{v}}$. One usually defines three unit vectors along each of the coordinate axes of the frame of reference:

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad \hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$



A Vector's Components Change In Different Coordinate Systems

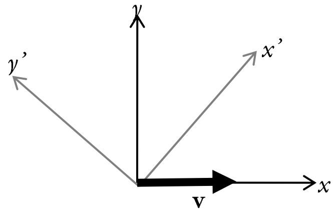
It's important to realize that a vector is a geometrical quantity which **remains unchanged** as we transform to a different coordinate system. What will change are its components (i.e. its projections on the axes of the new system):



$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{\text{xyz frame}}$$

$$\mathbf{v} = \begin{pmatrix} v_x' \\ v_y' \\ v_z' \end{pmatrix}_{\text{x'y'z' frame}}$$

For example, consider the 2D unit-sized vector **v** in the following illustration and its components in the two coordinates systems xyz and x'y'z':



$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\substack{xy \\ \text{frame}}}$$

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\substack{x'y' \\ \text{frame}}}$$

In both examples the vector has unit magnitude, since magnitude is a geometrical construct which does not depend on the frame of reference. For example, in the x'y' frame,

$$|\mathbf{v}| = \sqrt{(v_x')^2 + (v_y')^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1 .$$

However, in the x'y' frame, it has non-zero components along both the x' and y' axes.

Usually it is obvious from the context which frame of reference we are discussing (if there is even more than one), so usually the frame of reference is not mentioned over and over again.

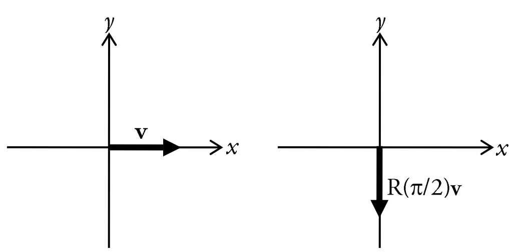
Rotations

Rotation Matrices

A simple 2D rotation matrix in the 2D plane is:

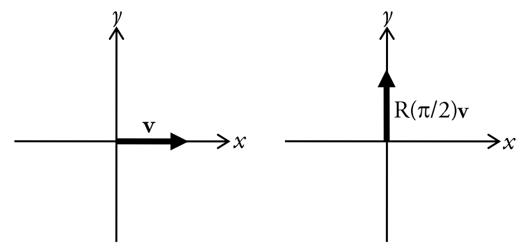
$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad (\text{clockwise}) .$$

If we apply R(θ) to a vector **v** we will rotate it by an angle θ **clockwise**. For example, if **v**=(1,0) starts out along the x-axis, then R(π/2)**v**=(0,-1) is rotated by 90° clockwise and points along the –y axis.



One can equally construct a counter-clockwise rotation by simply taking θ→-θ:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (\text{c.c.})$$

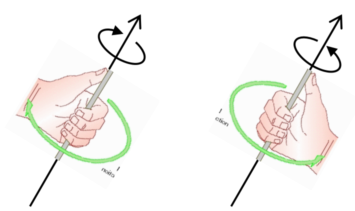


It is important to state clearly whether a 2D rotation is clockwise or anti-clockwise.

A 3D rotation is specified via three quantities:

1. The **axis** of rotation, given by a unit vector $\hat{\mathbf{n}} = (n_x, n_y, n_z)$.
2. The angle of rotation, θ.
3. Whether the rotation is right-handed or left handed.

A rotation is **right handed** if, when you place the thumb of your right hand along the axis of rotation and curl your fingers, the fingers curl in the direction of rotation. Likewise, a rotation is **left handed** if, when you place the thumb of your left hand along the axis of rotation and curl your fingers, the fingers curl in the direction of rotation.



Left handed rotation

Right handed rotation

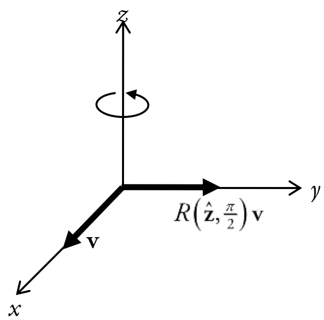
The general right-handed 3D rotation matrix is:

$$R(\hat{\mathbf{n}},\theta) = \begin{pmatrix} c\theta + n_x^2(1-c\theta) & n_x n_y(1-c\theta) - n_z s\theta & n_x n_z(1-c\theta) + n_y s\theta \\ n_y n_x(1-c\theta) + n_z s\theta & c\theta + n_y^2(1-c\theta) & n_y n_z(1-c\theta) - n_x s\theta \\ n_x n_z(1-c\theta) - n_y s\theta & n_z n_y(1-c\theta) + n_x s\theta & c\theta + n_z^2(1-c\theta) \end{pmatrix}$$

where cθ≡cosθ, sθ≡sinθ. For example, a right handed rotation about the z-axis by an angle θ is obtained by setting $\hat{\mathbf{n}} = (0,0,1) = \hat{\mathbf{z}}$:

$$R(\hat{\mathbf{z}},\theta)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\quad (\text{R.H. rotation})$$

For example, if **v**=(1,0,0) points along the x-axis, then $R(\hat{\mathbf{z}},\frac{\pi}{2})\mathbf{v}=(0,1,0)$ points along the y-axis:



Rotation At A Constant Angular Velocity

A vector rotating at a constant angular velocity increments its angle (around the rotation axis) at a linear rate, so

$$\theta(t)=\omega t\ .$$

The constant of proportionality, ω, is called its *angular velocity* and has units of radians per unit time. For example,

$$\omega=4\pi\cdot\tfrac{\text{rad}}{\text{ms}}$$

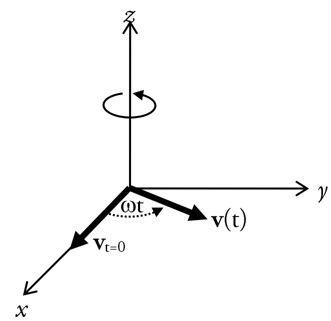
means the vector completes two circles in one millisecond.

Let's take a concrete example. If we start out along the *x*-axis at time t=0, then:

$$\mathbf{v}_{t=0}=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

It this vector rotates at a constant angular velocity ω around the *z*-axis (a RH rotation), we can write

$$\begin{aligned}\mathbf{v}(t)&=R(\hat{\mathbf{z}},\omega t)\,\mathbf{v}_{t=0}\\&=\begin{pmatrix}\cos\omega t & -\sin\omega t & 0\\ \sin\omega t & \cos\omega t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\\&=\begin{pmatrix}\cos\omega t\\ \sin\omega t\\ 0\end{pmatrix}\end{aligned}$$



Complex Numbers

A Complex Number Has A Real And Imaginary Part

A **complex number** is a number of the form $z=x+iy$, where x and y are real numbers and $i=\sqrt{-1}$. For example, z=4, z=5i and z=-3+2i are all complex numbers.

Complex numbers are added "component-wise", so

$$\begin{aligned}z_1+z_2&=(x_1+iy_1)+(x_2+iy_2)\\&=(x_1+x_2)+i(y_1+y_2)\end{aligned}$$

They can also be multiplied:

$$\begin{aligned}z_1z_2&=(x_1+iy_1)(x_2+iy_2)\\&=x_1x_2+iy_1x_2+ix_1y_2+i^2y_1y_2\end{aligned}$$

Since $i=\sqrt{-1}$, then $i^2=-1$ and

$$z_1z_2=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)\,.$$

The **complex conjugate** of a complex number *z* =(*x*+*iy*) is denoted either with a bar $\bar{z}$ or an asterisk z^*, and obtained by replacing *i* with -*i*:

$$\bar{z} = x - iy .$$

The magnitude of a complex number is defined in a similar manner to that of a vector:

$$|z| = \sqrt{x^2 + y^2} .$$

Note that

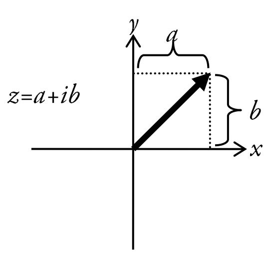
$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 + y^2 = |z|^2 .$$

Two complex numbers are equal to each other if and only if their real and imaginary components are equal. So, if z_1 $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$, and $z_1 = z_2$, then $x_1=x_2$ and $y_1=y_2$.

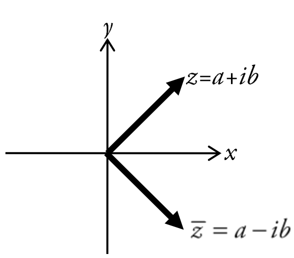
Complex numbers have some pitfalls you should watch out for. For example, $|z|^2 \neq z^2$, which can be seen by taking $z=i$, for which $|z|^2 = 1$ but $z^2=-1$.

Complex Numbers As Vectors

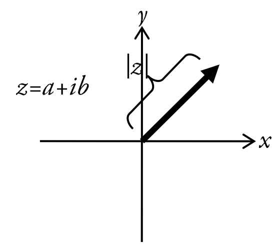
We can think of complex numbers as vectors in the 2D *xy*-plane:



This geometric interpretation lets us assign meaning to some of the quantities we spoke of before. For example, conjugation can be thought of as mirroring the number about the x-axis, since we flip its imaginary component:



The magnitude of the vector is simply the magnitude of *z*:



Euler's Identity

Euler's identity says that

$$e^{ix} = \cos(x) + i\sin(x) .$$

It's very very useful in many fields of applied mathematics. First, a nice trick:

$$e^{-i\pi} + 1 = 0$$

This involves all of the most important notions in mathematics in one line (e, π, 1, 0, +, i, = and *i*). You can also use Euler's identity to prove all sorts of trigonometric identities. For example, note that:

$$e^{2ix} = e^{ix}e^{ix} .$$

Use Euler's identity on both sides and obtain:

$$\cos(2x) + i\sin(2x)$$
$$= \cos^2(x) - \sin^2(x) + i2\cos(x)\sin(x)$$

Equating real and imaginary components gives two famous trigonometric identities:

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\sin(2x) = 2\cos(x)\sin(x)$$

Just as we can go from $e^{i\theta}$ to sines and cosines, we can also go the opposite way:

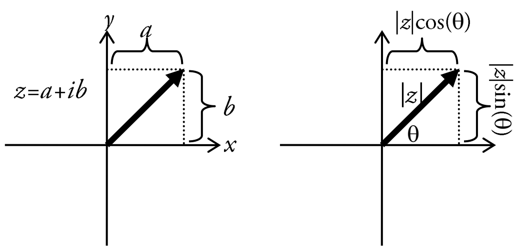
$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos x - i\sin x$$

We can solve for cos(x) and sin(x) by adding and subtracting, obtaining:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Magnitude And Phase

If we denote by θ the angle made by the complex number z and the x-axis, we can use basic trigonometry to write it as:



$$\begin{aligned} z &= a + ib \\ &= |z|\cos\theta + i|z|\sin\theta \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$

We can rewrite the last line using Euler's identity:

$$z = |z| e^{i\theta}$$