Lecture 4
Excitation & Acquisition
Lecture Notes by Assaf Tal

Now that we have discussed the physical principles by which spins and magnetic fields interact, we come to the main question: how **do** we measure a signal from a sample containing spins? More precisely, we want to create an image of our body, which – to a first approximation – means mapping the density of water molecules as a function of position. Because the bulk magnetization per unit volume is proportional to the density of water at each point, we can rephrase our question and ask: **How can we map the bulk magnetization per unit volume, M(r)?** The answer is that we need to first **excite** the spins, **encode** their positions somehow, and then **acquire** a signal from them and **reconstruct** it. The physics of excitation and acquisition will be covered in this lecture. Encoding will be covered in the next lecture, and reconstruction in the following lecture.

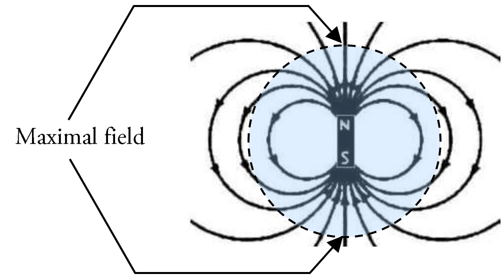
To Measure a Signal, Spins Must Be Excited

The Static Nuclear Magnetic Field is Too Weak to be Reliably Detected

We've previously calculated the bulk magnetic moment of 1 cm^3 of water in a 3 T magnetic field and found it to be about $|\mathbf{M}| \sim 10^{-8}$ J/T. Imagine this voxel is inside the human body and must be detected in a coil wrapped around the body – say, 20 cm away from it. The magnetic field of the voxel is dipolar

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{M}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{M}}{r^3}$$

If we consider a sphere of radius r around **M**, the field will be maximal at the poles:



where its value will be (putting r=0.2 m, $M=10^{-8}$ J/T, $\mu_0=4\pi\cdot 10^{-7}$ N/A^2):

$$|\mathbf{B}\ \mathbf{r}| = \frac{\mu_0 M}{2\pi r^3} \sim 10^{-13}\ T = 0.1\ pT.$$

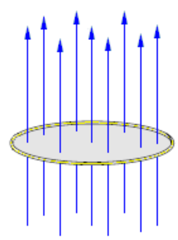
This is an incredibly small magnetic field. The extremely weak magnetic fields produced by neuronal currents are 1-2 orders of magnitude larger, and even those require extremely specialized hardware based on super-cooled magnetic field detectors known as SQUIDs. SQUIDs are used in *magnetoencephalography* (MEG) machines, but they are not sensitive enough to detect the sort of signals we're interested in. It is possible to construct sensitive enough detectors, but even if we put aside the engineering complexity and cost of these in addition to the MRI magnet itself, we are still faced with further problems:

- How can we separate the tiny field created by the nuclear spins from the much larger sources of magnetic fields created by other phenomena in the body, e.g. neuronal currents, or the membrane potentials in the heart?
- How can we deduce the distribution of spins **within** the body by detecting the magnetic field **outside** the body? That is, can we **image M(r)** inside the body by measuring **B**(**r**) outside the body? These sort of problems are known as **inverse problems** and are often very difficult to solve properly (the same issue plagues MEG).

Is there a better way to detect the MRI signal? The answer is yes, and it is linked to the phenomenon of **resonance** and the precession of spins around the main B_0 field. To see why precession makes it easy for us to measure the magnetic fields of nuclear magnetic moments, we need to first discuss signal acquisition.

Time Varying Magnetic Fields Can Be Picked Up with A Coil: Faraday's Law

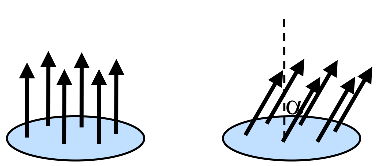
The **magnetic flux** through a coil equals the integral of the normal component of the magnetic field through the surface of a coil:



Mathematically, this amounts to a surface integral over the surface enclosed by the loop:

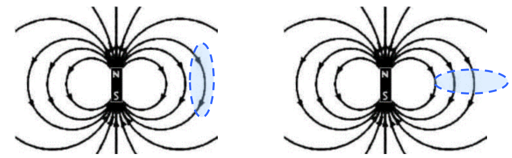
$$\phi = \int \mathbf{B} \cdot d\mathbf{S}\,.$$

Intuitively, this is the "amount of magnetic field lines crossing the coil." For example, if we had a constant magnetic field **B** normal to the coil, and the coil had area A, the magnetic flux through it would be A·B. If **B** were to make an angle α with the normal to the coil's surface, the flux would be reduced to A·B·cos(α):



Flux: A·B   Flux:

Another example: consider placing a coil around a magnetic moment. In one orientation there would be no flux through the coil, while if we were to rotate the coil by 90° the flux would be maximal:



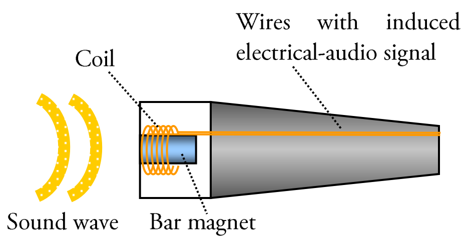
No flux   Maximal

The importance of flux comes from **Faraday's law**:

> A time varying flux ϕ(t) through a coil will generate a voltage given by:
>
> $$v = -\frac{d\phi}{dt}$$
>
> (Faraday's Law)
>
> This is a very different method of detection compared to optics, where we pick up photons emitted by radiating molecules. Radiation is a far-field phenomenon, while induction (Faraday's Law) is a near-field phenomenon.

Note that the generated voltage is not proportional to the amount of flux (ϕ), but rather to its time derivative. Even if ϕ is large it will not generate any current if it is static.

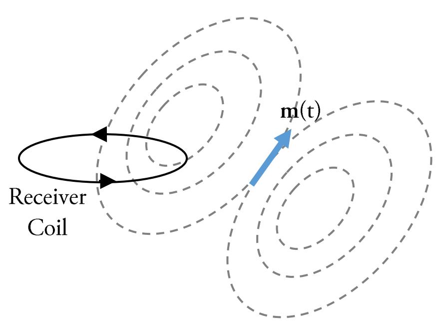
This law underlies much of modern electricity and electronics, since it provides a mechanism for turning one type of energy into another. An example is the microphone: some microphones, known as dynamic microphones, are comprised of a diaphragm connected to a bar magnet, around which a coil is tightly wound. As sound waves oscillate the diaphragm they also physically move the magnet which changes the magnetic field's flux through the coil as a function of time. These oscillations are therefore reproduced in the electrical signal induced in the coil and recorded on tape (or, in modern hardware, on the computer):



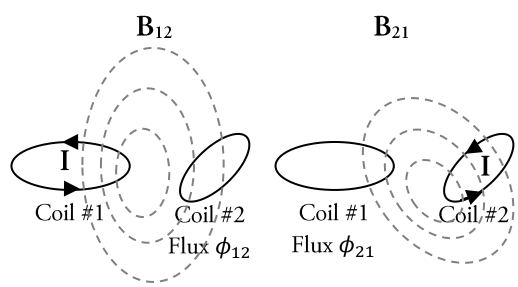
In our case, a precessing magnetic moment will create a precessing dipolar field around it – that is, a time-varying magnetic field. The dipolar field will rotate at the same angular velocity as the spin. A current will then be generated in a suitably-positioned coil, known as a **receiver coil**. Any receiver coil can also create a magnetic RF field by putting an oscillating current through it, making it a **transmitter coil**. Thus, any coil can be used for both reception and transmission (but not simultaneously).

Deriving a Simpler Expression for the Induced Voltage in a Coil: The Law of Reciprocity

Imagine trying to calculate the magnetic flux of a time varying magnetic moment **m**(t) through a coil:



You would have to write down the time-dependent dipole field and integrate it over the surface of the coil. The coordinate system might be tilted and in general the calculation of the induced voltage using Faraday's law directly will be difficult. To overcome this, we will make use of a very neat trick known as the **principle of reciprocity**, which will enable us to derive a simple expression for the flux in the receiver coil due to the time varying magnetic moment. This principle can be stated very simply (although its proof, which is difficult, will be omitted): If we take any two coils, then the flux through coil #2 created by putting a current I through coil #1 will be equal to the flux through coil #1 created by putting the same current I through coil #2. Graphically, if we image two configurations,

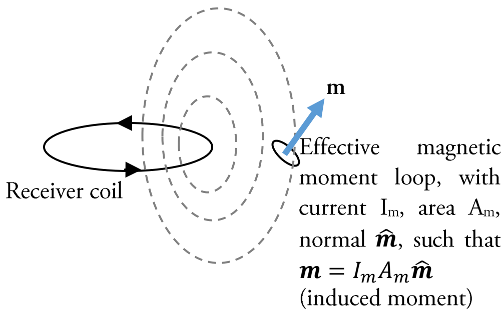


then the principle of reciprocity tells us that

$$\phi_{12} = \phi_{21}$$

This is quite surprising at first glance, but it's a very useful identity. Proving the principle of reciprocity is beyond the scope of this class, but these sorts of principles appear throughout classical electrodynamics. Simply put, reciprocity tells us that **a good receiver is a good transmitter**: if coil #1 is meant to pick up a field put out by coil #2, its response (the flux through it) will be given by the flux that it will create in coil #2 when you put the same current through it.

Now let's apply this principle to our problem and "turn it on its head" in a sense. First, we can model the microscopic nuclear magnetic moment using an infinitesimal loop of current, since we remarked such a loop will create a magnetic moment **m**=(area)×(current):



There is no requirement for the moment's loop to be co-planar with the receiver coil, nor do we assume the receiver coil is planar (it's just easier for me to draw a planar one).

We're interested in calculating the flux through the receiver coil due to the current I_m in the "imaginary" magnetic moment loop. The principle of reciprocity tells us that this is equal to the flux through the imaginary magnetic loop coil due to a current I_m through the receiver coil!

Let's denote by $\boldsymbol{B}_{rec}(\boldsymbol{r}, t)$ the field created by putting a current I_m through the receiver coil. First, note that the field created by a coil should scale by the current you put through it: doubling the current will double the magnetic field vector in each point in space, so we can write

$$\boldsymbol{B}_{rec}(\boldsymbol{r}, t) = I_m(t)\widehat{\boldsymbol{B}}_{rec}(\boldsymbol{r})$$

where $\widehat{\boldsymbol{B}}_{rec}(\boldsymbol{r})$ is the field generated by a unit current in the receiver coil. Because the magnetic moment's loop is so small, we can assume $\mathbf{B}_{rec}$ is effectively constant across it, so we can approximate the flux through it:

$$\phi_{moment} = \boldsymbol{B}_{rec}(\boldsymbol{r}, t) \cdot (A_m \widehat{\boldsymbol{m}})$$

where $A_m \widehat{\boldsymbol{m}}$ is a vector perpendicular to the plane and having a magnitude equal to the area of the loop. Rearranging a bit,

$$\begin{aligned}\phi_{moment} &= \widehat{\boldsymbol{B}}_{rec}(\boldsymbol{r}) \cdot (A_m I_m \widehat{\boldsymbol{m}}) \\ &= \widehat{\boldsymbol{B}}_{rec}(\boldsymbol{r}) \cdot \boldsymbol{m}(t)\end{aligned}$$

The principle of reciprocity tells us that this flux equals the flux created by the moment through the receiver coil!

$$\phi_{rec} = \phi_{moment}$$

Now, all that remains to do is differentiate it with respect to time to obtain the induced voltage in the coil:

$$v_{rec} = -\widehat{\boldsymbol{B}}_{rec}\ \boldsymbol{r}\ \cdot \tfrac{d\boldsymbol{m}}{dt},$$

The above expression can be extended to a spatial distribution of moments by integrating over space:

$$v_{rec} = -\int_{body} \widehat{\boldsymbol{B}}_{rec}\ \boldsymbol{r}\ \cdot \frac{d\boldsymbol{M}\ \boldsymbol{r}, t}{dt} dV$$

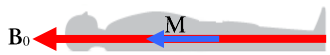
Precessing Spins Induce a Measurable Currents in the Receiver Coils

We are now in a position to calculate the voltage induced in a simple receiver coil due to a precessing spin and show that it is significant enough to be detectable.

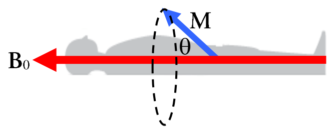
As we've remarked previously, any spin will precess about a constant magnetic field. In particular, if we place the spins in a static, strong magnetic field – say, the 3 Tesla field of a typical MRI scanner – it will precess. For a hydrogen nucleus, this precession frequency would be

$$\nu_0 = \bar{\gamma} B_0 = 42.57 \tfrac{\text{kHz}}{\text{mT}} \cdot 3\ \text{T} \approx 127\ \text{MHz}.$$

Or $\omega_0 = \gamma B_0 = 2\pi\nu_0$. Using a coil and the law of reciprocity we can measure the time-dependent flux induced by the spin. This dynamic measurement will create a significantly larger signal than a static one. We are, however, faced with a paradox: at thermal equilibrium, the bulk magnetic moment is parallel to B_0, and hence the precession is "degenerate" – $\mathbf{M}$ remains static (even though the microscopic moments, $\mathbf{m}$, will precess). For us to observe true precession, **M** must make some non-zero angle with B_0:

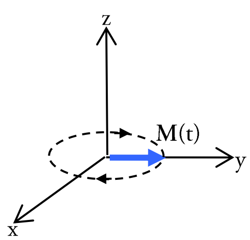


At thermal equilibrium, **M** and $\mathbf{B_0}$ are colinear, and no precession is observed.



Only by creating some nonzero angle θ between **M** and $\mathbf{B_0}$ – that is, only by **exciting M** - can we observe precession.

Creating such an angle is called **excitation**. Putting that aside for a moment, let's calculate the voltage induced in a coil put around the precessing spin via Faraday's law and the law of reciprocity. Here we take a wo circular coils of radius R in the xz and yz planes, with a point magnetic moment placed at the origin and performing some rotation as a function of time:



The components of the precessing spin have a sinusoidal time dependence:

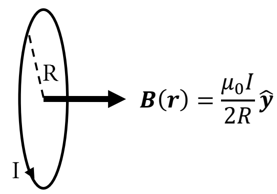
$$\boldsymbol{M}\ t\ = \begin{pmatrix} M \cdot \cos\ \omega_0 t \\ -M \cdot \sin\ \omega_0 t \\ 0 \end{pmatrix}.$$

(LH rotation about z)

Remember that the law of reciprocity tells us that the voltage induced in the coil can be calculated via

$$v_{rec} = -\widehat{\boldsymbol{B}}_{rec}\ \boldsymbol{r}\ \cdot \frac{d\boldsymbol{m}}{dt},$$

where $\widehat{\boldsymbol{B}}_{rec}\ \boldsymbol{r}$ is the field created by a unit current the loop at the position of the magnetic moment (at the origin). The expression for the magnetic field created by a loop of current at its center is well known from basic magnetism:



where R is the ring's radius, I the current, and $\hat{\boldsymbol{y}}$ a unit vector normal to the plane of the ring. The reason we placed the loop in the xz-plane is to maximize the magnetic flux through it (had we placed it in the xy plane there would have been no flux through it!). We take unit current (I=1), so the field created by the receiver at the position of the magnetic moment at its center is:

$$\widehat{\boldsymbol{B}}_{rec}(\boldsymbol{r}) = \frac{\mu_0}{2R}\hat{\boldsymbol{y}}$$

and so:

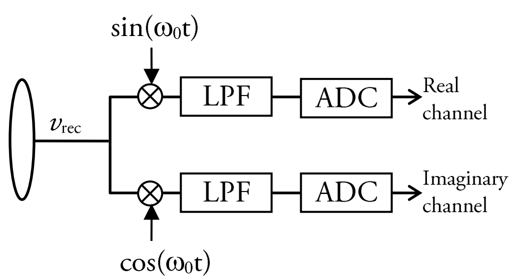
$$\begin{aligned} v_{rec} &= -\frac{d}{dt}\left(\frac{\mu_0}{2R}\boldsymbol{y}\cdot\boldsymbol{M}\right) \\ &= -\frac{\mu_0}{2R}\frac{dM_y}{dt} \\ &= -\frac{M_0\mu_0\omega_0}{2R}\cos\ \omega_0 t \end{aligned}$$

<u>**Number Time**</u>. For $\mu_0 = 4\pi \times 10^{-7}$ V·s/(A·m), $\omega = 2\pi \cdot 127$ MHz for protons at 3 Tesla, R = 15 cm (head coil), and $M_0 = 10^{-8}$ J/T (previously calculated magnetization of 1 mL of water), we get 30 μV, around the right order of magnitude for the voltages detected in magnetic resonance.

This is a small but detectable voltage level with today's electronics, and this is the basis of signal reception in modern MRI. The smaller the radius of the coils, R, the better: always build coils that are as small as possible! Furthermore, the signal is proportional to $\omega_0 = \gamma B_0$, and increases with B_0 (although an exact analysis of the SNR will await a later chapter).

Some Additional Concepts in Signal Detection: "Heterodyne" Detection

While we've covered the basic physical principles behind signal detection, there are some things we will only mention here qualitatively. The following diagram describes a typical cascade of actions that are applied to the receiver coil's voltage:



1. The acquired signal needs to be converted to a digital signal via an **analog to digital converter** (ADC).
2. Because such ADCs don't always handle high frequencies efficiently, such as the Larmor precession at ~100 MHz, the signal must first be **down-converted**, meaning it must be taken from a ~100 MHz signal to the kHz range.
3. Another thing that is done along the way is splitting the signal, and adding a 90° phase to the second signal. The two signals are combined inside the computer (after the ADC) into one complex signal. This effectively turns a sine or cosine real signal into a complex exponential one:

$$\cos(\omega_0 t) \rightarrow e^{-\iota\omega_0 t}$$

The Appendix discusses these steps in greater detail and how they relate to signal detection. We will have some things to add to that towards the end, after we discuss excitation.

The Spins Can Be Excited with a Resonant RF Field

A Small RF Field Can Have a Large Effect if it is Resonant

The discussion of the previous section has shown that, in order to induce non-zero voltage, we must tilt the magnetization vector away from equilibrium and have it precess. Indeed, the basic MR experiment can be described as follows:

- **Thermal Equilibrium**: At thermal equilibrium, the spins are aligned along B_0 and do not precess.
- **Excitation**: The spins are somehow **excited**, that is, tilted to some angle θ with respect to B_0. This usually happens quickly and relaxation can be neglected. The reason this is called an **excitation** is because energy needs to be put into the system: the energy of a moment **M** making an angle θ with $\mathbf{B}_0$ is $E = -MB_0\cos(\theta)$, and is lowest when $\theta = 0°$ and increases as θ increases (up until $\theta = 180°$).
- **Precession & Detection**: Once tilted, they precess and give off a time dependent magnetic field. The magnetic field induces a voltage in a nearby RF coil via Faraday's law. We can also further manipulate the spins with magnetic fields during this period to bring out particular contrast types. We usually have a time ~ T_2 before decoherence "eats up" the observable precessing magnetization.
- **Thermalization**: Relaxation processes kick in. The transverse magnetization decays with a time constant T_2 while the longitudinal magnetization builds up back up due to T_1 relaxation. If we wait for a

time ≈ 5·T_1, the magnetization will be back at its thermal equilibrium value.

Each such block (excite-acquire-wait) is called a **scan**. It is in fact not mandatory to wait for a time 5·T_1 for the spins to return to thermal equilibrium; we'll see later on that waiting a shorter amount of time has both benefits (shorter scan times) and disadvantages (less signal per scan). For now, however, we'll assume that is the case, so M is equal to M_0 and points along the z-axis before the beginning of each scan.

We've already remarked that $B_{RF} << B_0$. Typical RF field strengths are ~10 μT, while $B_0 > 1$ T. How can we hope to non-negligibly excite the spins with such a weak, seemingly insignificant RF field? The answer is that we can use a **resonant field** that oscillates at the Larmor frequency (indeed, the **R** in **MRI** stands for **resonance**).

$\omega_F = \omega_0$, and decrease substantially as $\omega_F \gg \omega_0$ (or $\omega_F \ll \omega_0$).

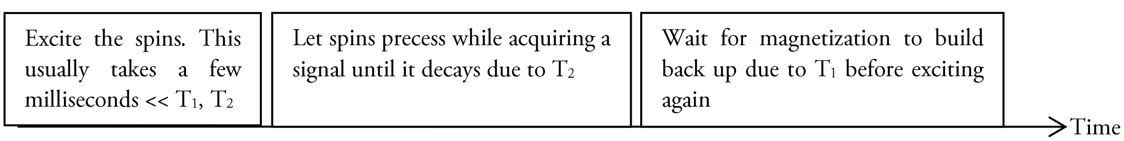
How does resonance work? Briefly, the system returns to the same state with a period of ω_0, and if the maximal applied force is repeatedly applied at the same state it can produce a "coherent" buildup effect, adding up constructively between visits of the system to the same state.

In our case, to tilt the nuclear magnetic moment effectively away from B_0, we will apply a resonant external radiofrequency field, oscillating at its natural (Larmor) frequency. Namely, we are going to solve the Bloch equations setting **G**=0, and

$$\boldsymbol{B}_{RF}(t) = B_1 \cos(\omega_{RF} t)\hat{\boldsymbol{x}} - B_1 \sin(\omega_{RF} t)\hat{\boldsymbol{y}}$$

with

Schematic for a simple single experiment ("scan")
(not drawn to temporal scale)



Resonance refers to the fact that many physical systems have a "natural frequency" (or set of frequencies) associated with them, and that applying an external "force" which oscillates at the natural frequency of the system produces an extraordinarily large response compared to a force which oscillates at a "far away" frequency". For example, a mass on a spring, if displaced from equilibrium and left to its own devices, will oscillate with a frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$

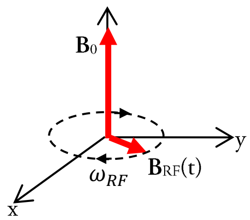
If we now hold the end of the spring and shake it with a frequency ω_F, the resulting amplitude of the oscillating mass will be maximal when

$$\omega_{RF} = \omega_0 \quad (\text{"on-resonance irradiation"})$$

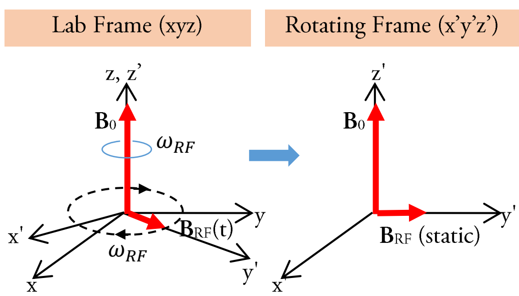
This means we will need to solve the Bloch equations with a time dependent magnetic field. Although a numerical solution is possible, we will employ a frame transformation trick which will enable us to solve this problem analytically.

Simplifying the Problem: Transforming to a Frame Which Rotates at the Same Frequency as the RF Field Makes it Appear Static (The Rotating Frame)

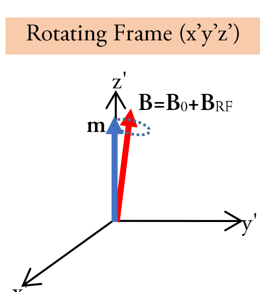
To make the problem easier, we will consider the magnetic field in a frame of reference which rotates in such a manner so as to make it appear static. In other words, if the RF field rotates in the xy-plane with an angular velocity ω_{RF},



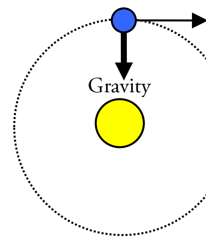
then, by transforming to a new set of axes, (x'y'z'), which rotates around the z axis with the same angular velocity ω_{RF}, we render $\mathbf{B}_{RF}$ constant in that frame:



This seems like a great way for simplifying the problem, but it creates a problem of its own: **applying the Bloch equations naively in this rotating frame will yield incorrect/unphysical results**. For example, in the rotating frame as presented above, the total magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_{RF}$ is a static vector that overwhelmingly along the z-axis (because B_0~Tesla, B_{RF}~μT). If the magnetization will start along the z-axis, it will barely precess about the total magnetic field and not be excited at all:

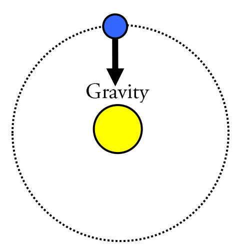


We **know** this from experiments to be untrue. Before I show you how to fix this, let me give you an analogy from mechanics that will help you understand the issue. Imagine the earth going around the sun in a circle:

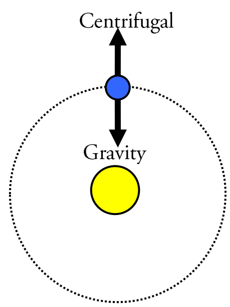


This can be understood by an observer in space the following way: the Earth wants to "go forward" but gravity pulls it "inward", curving its path into a circle. In effect, the Earth is continuously "falling" into the sun, but escaping doom thanks to its tangential velocity. All this is all a consequence of Newton's second law, F=ma.

Next, imagine how things would look to an observer standing on the sun and rotating with it. Neglecting for the time being the weather on the surface, the Earth would appear stationary to such an observer:



If that observer would try to use Newton's law F=ma to understand his world he would fail: according to $F=F_{gravity}=ma$, earth should be falling towards the sun, but it isn't! The truth is that when you transform to a rotating frame you need to add a **fictitious force**. That is, you need to pre-suppose a force which doesn't arise out of any physical source, called the centrifugal force, to explain how it is possible for the earth to remain stationary:



So, in mechanics when you try to understand things in a rotating frame you need to do two things:

1. Understand how things in the "real" frame would look in the rotating frame (e.g., the Earth would remain still).
2. Add fictitious forces (e.g., the centripetal force).

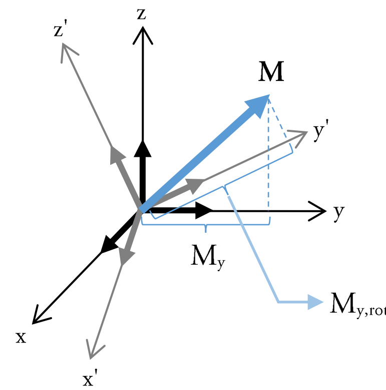
A similar thing happens when you go to a rotating frame in magnetic resonance, rotating with the same angular velocity as the RF field:

1. First, the RF field needs to be transformed to the rotating frame (we've done this).
2. Next, we will need to add a fictitious "force" - field, to be precise - given by $\boldsymbol{B}_{fict} = -\boldsymbol{\omega}_{rot}/\gamma$, to make the Bloch equations work in the noninertial rotating frame. Here, $\boldsymbol{\omega}_{rot}$ is a vector point along the axis of rotation with a magnitude equal to the angular velocity of the rotation (we've taken $|\boldsymbol{\omega}_{rot}| = \omega_{RF}$).

Rewriting the Bloch Equations in the Rotating Frame: The Effective Field

Before tackling the full problem of frame transformations, let's talk about vectors and how to represent them in different frames.

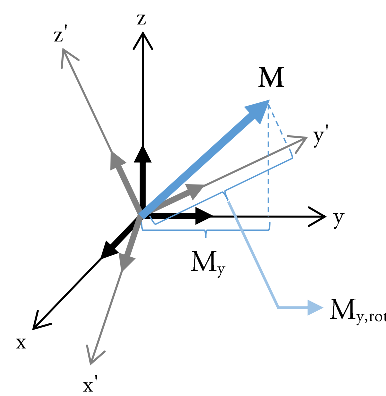
Imagine a vector **M** and two frames of references – a static (laboratory) frame with time independent, fixed unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (black), and a rotating frame with unit vectors $\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}'$ (gray):



You can express the vector **M** in terms of the unit vectors in each of the coordinate systems:

$$\boldsymbol{M}(t) = M_x\hat{\boldsymbol{x}} + M_y\hat{\boldsymbol{y}} + M_z\hat{\boldsymbol{z}}$$
$$= M_{x,rot}\hat{\boldsymbol{x}}' + M_{y,rot}\hat{\boldsymbol{y}}' + M_{z,rot}\hat{\boldsymbol{z}}'$$

The difference between these components is illustrated below for the y-component of **M**(t) in the original (xyz) and rotating (x'y'z') frame:



Another thing to keep in mind is that $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are static and their time derivatives are zero, while $\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}'$ are all time-dependent with non-zero time derivatives.

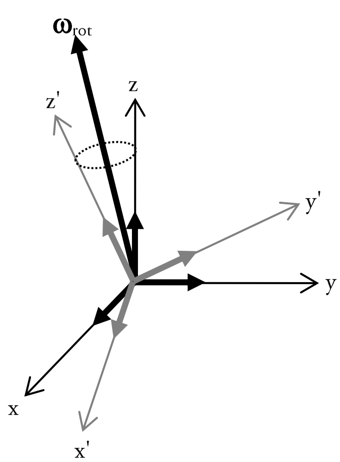
The time derivative of **M** can be written using the expression $\boldsymbol{M}(t) = M_{x,rot}\hat{\boldsymbol{x}}' + M_{y,rot}\hat{\boldsymbol{y}}' + M_{z,rot}\hat{\boldsymbol{z}}'$ and the chain rule as:

$$\frac{d\boldsymbol{M}}{dt} = \frac{dM_{x,rot}}{dt}\hat{\boldsymbol{x}} + \frac{dM_{y,rot}}{dt}\boldsymbol{y}' + \frac{dM_{z,rot}}{dt}\hat{\boldsymbol{z}}'$$
$$+M_{x,rot}\frac{d\hat{\boldsymbol{x}}'}{dt} + M_{y,rot}\frac{d\hat{\boldsymbol{y}}'}{dt} + M_{z,rot}\frac{d\hat{\boldsymbol{z}}'}{dt}$$

simply because the unit vectors themselves change with time. The first three components describe the time derivative of the components of **M** as they appear to change **to an observer in the x'y'z' frame**. The last three components have to do with the frame's rotation. What we're after is an equation of motion, analogous to the Bloch equations, that will let us solve for the components of **M** as they appear in the rotating frame:

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} \equiv \frac{dM_{x,rot}}{dt}\hat{\boldsymbol{x}}' + \frac{dM_{y,rot}}{dt}\hat{\boldsymbol{y}}' + \frac{dM_{z,rot}}{dt}\hat{\boldsymbol{z}}'$$

Let us introduce a vector $\boldsymbol{\omega}_{rot}(t)$, that can change with time, such that its direction describes the axis of rotation and its magnitude the angular velocity of the rotation (in the left hand sense):



If the rotating frame is rotating with an angular velocity ω_{rot} about an axis given by the unit vector $\hat{\mathbf{n}}$, then each of the axes of the rotating frame precess about the vector $\boldsymbol{\omega}_{rot}=\omega_{rot}\hat{\mathbf{n}}$. This means each obeys a precession equation identical (formally) to the Bloch equation:

$$\frac{d\boldsymbol{x}'}{dt} = \boldsymbol{x}' \times \boldsymbol{\omega}_{rot}$$
$$\frac{d\boldsymbol{y}'}{dt} = \boldsymbol{y}' \times \boldsymbol{\omega}_{rot}$$
$$\frac{d\boldsymbol{z}'}{dt} = \boldsymbol{z}' \times \boldsymbol{\omega}_{rot}$$

In this analogy,

$$\hat{\boldsymbol{x}}' \sim \boldsymbol{M}$$
$$\boldsymbol{\omega}_{rot} \sim \gamma \boldsymbol{B}$$

Using this,

$$\begin{aligned}\frac{d\boldsymbol{M}}{dt} &= \left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} + M_{x,rot}\frac{d\hat{\boldsymbol{x}}'}{dt} + M_{y,rot}\frac{d\hat{\boldsymbol{y}}'}{dt} + M_{z,rot}\frac{d\hat{\boldsymbol{z}}'}{dt} \\ &= \left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} + \boldsymbol{\omega}_{rot} \times \left(M_{x,rot}\hat{\boldsymbol{x}}' + M_{y,rot}\hat{\boldsymbol{y}}' + M_{z,rot}\hat{\boldsymbol{z}}'\right) \\ &= \left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} + \boldsymbol{\omega}_{rot} \times \boldsymbol{M}(t)\end{aligned}$$

The left hand side of this equation equals $\gamma\boldsymbol{M}(t) \times \boldsymbol{B}(t)$ by virtue of the Bloch equation. Plugging in and rearranging, we obtain

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = \gamma\boldsymbol{M} \times \left(\boldsymbol{B} - \frac{\boldsymbol{\omega}_{rot}(t)}{\gamma}\right)$$

This is precisely the Bloch equation but with an effective field $\mathbf{B}_{eff} = \mathbf{B} - \frac{1}{\gamma}\boldsymbol{\omega}_{rot}$. The additional term $\boldsymbol{B}_{fict} = -\boldsymbol{\omega}_{rot}/\gamma$ is called the fictitious field.

We are free to express either side in either frame of reference, but should remember that $\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot}$ has a simple physical interpretation in the x'y'z' rotating frame (having components equal to the time derivatives of **M** as it appears in the rotating frame), and should therefore express all quantities in the rotating frame. For example, if **B**(t) is

$$\begin{aligned}\boldsymbol{B}_{RF}(t) &= B_1\cos(\omega_{RF}t)\,\hat{\boldsymbol{x}} - B_1\sin(\omega_{RF}t)\,\hat{\boldsymbol{y}} \\ &= B_1\hat{\boldsymbol{x}}'\end{aligned}$$

it makes more sense to use the second expression so we can equate the components of both sides of our vector equation in the rotating frame.

The above equation is true for any rotating frame. However, in MRI, when we speak of "**the**" rotating frame, we will be referring to a frame which rotates at a constant angular velocity $\omega_{rot} = \omega_{RF}$ about the z-axis according to the left-hand rule:

> "The" **rotating frame** is one which rotates with a constant angular frequency with the RF field: $\omega_{rot} = \omega_{RF}$ (left hand rule). For "the" rotating frame: $\boldsymbol{\omega}_{rot} = \omega_{rot}\hat{\mathbf{z}} = \omega_{RF}\hat{\mathbf{z}}$.

When expressed in the rotating frame, the components of the effective field $\boldsymbol{B}_{eff} = \boldsymbol{B} - \frac{1}{\gamma}\boldsymbol{\omega}_{rot}$ are:

$$\boldsymbol{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ B_0 - \frac{\omega_{rot}}{\gamma} \end{pmatrix}$$

(in the rotating frame: $\omega_{rot} = \omega_{RF}$)

If we further select $\omega_{RF} = \gamma B_0 = \omega_0$ we are **on resonance**: the RF irradiates the spins at the same frequency as their natural frequency, ω_0. In this case:

$$\boldsymbol{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}$$

On resonance: $\omega_{RF} = \omega_0$

In the rotating frame: $\omega_{rot} = \omega_{RF}$

If we select $\omega_{RF} \neq \omega_0$, we are **off resonance**. We can then define the **offset** $\Delta\omega = \omega_0 - \omega_{RF}$. The effective field is then:

$$\boldsymbol{B}_{\boldsymbol{eff}} = \begin{pmatrix} B_1 \\ 0 \\ \frac{\Delta\omega}{\gamma} \end{pmatrix}$$

Off resonance: $\omega_{RF} \neq \omega_0$

In the rotating frame: $\omega_{rot} = \omega_{RF}$

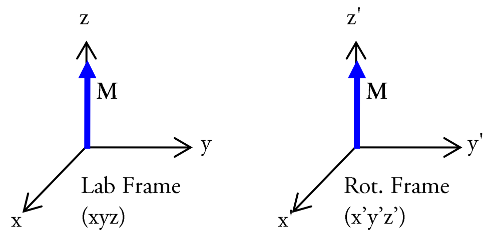
Clarifying Our Constants

At this point it is prudent to stop and summarize the different frequencies we use, their meanings and approximate magnitudes:

- $\omega_0 = \gamma B_0$ is on the order of 100 MHz at 3T. It is also called the "Larmor frequency" and represents the precession frequency of the spins around B_0, in the absence of gradients and RF fields.
- $\omega_1 = \gamma B_1$ is the magnitude of the external RF field. It is $\sim\mu T$ and is the size of the vector of the RF field, in both the lab and rotating frame.
- ω_{RF} is the frequency of the external RF field – i.e., how fast it rotates in the xy plane in the lab frame. It is usually close or equal to ω_0 (so $\omega_{RF} \sim \omega_0$).
- ω_{rot} is the rotation frequency of the rotating frame around the z-axis. It is always kept equal to ω_{RF} to "transform away" its time dependence and make it appear static in the rotating frame. Therefore, $\omega_{rot} = \omega_{RF} \sim \omega_0 \sim 100\ MHz$.
- $\frac{\Delta\omega}{\gamma} \equiv \Delta B$ is the z-component of the effective field. It is zero on-resonance, and non-zero off-resonance. It rarely exceeds a few hundred kHz.

The Bulk Magnetization Precesses Around the Effective Field in the Rotating Frame

We've seen the magnetization vector obeys the Bloch equations in the rotating frame, only swapping the field for an effective field, $\mathbf{B}_{eff} = \mathbf{B} - \frac{1}{\gamma}\boldsymbol{\omega}_{rot}$ and expressing that field in the rotating frame basis (i.e. as it would appear to an observer rotating with the frame). This means $\mathbf{M}$ precesses about $\mathbf{B}_{eff}$ in the rotating frame. Starting from thermal equilibrium at time t=0, $\mathbf{M}$ points along B_0 (taken to coincide with the z-axis) in both the laboratory and the rotating frames, which are also assumed to coincide for t=0:

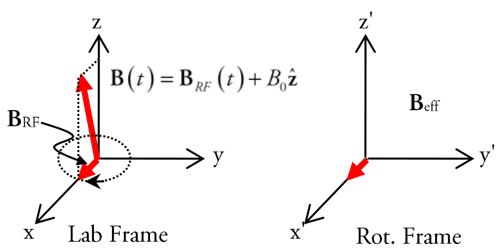


At time t=0 (thermal equilibrium), **M** points along the z-axis (same as the z' axis) in both frames.

Now we turn on the resonant RF field in the laboratory frame:

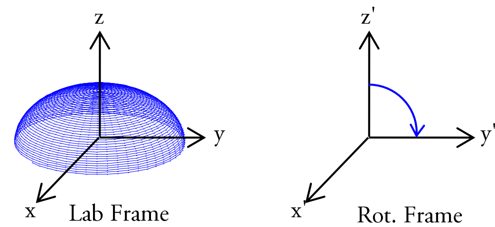
$$\boldsymbol{B}_{RF} = B_1 \cos(\omega_{RF} t)\hat{\boldsymbol{x}} - B_1 \sin(\omega_{RF} t)\hat{\boldsymbol{y}}$$

This field rotates in the xy-plane in the lab frame in the left-handed sense, and appears stationary in the rotating frame. Furthermore, if we assume our irradiation is on resonance, $\omega_{RF}=\omega_0$, the effective field in the rotating frame has no z-component:



The magnetic field **B** in the laboratory frame has a large z-component and a small, rotating xy-component (not shown to scale). In the rotating frame, assuming B_{RF} is on resonance ($\omega_{RF}=\omega_0=\gamma B_0$) the effective field is static.

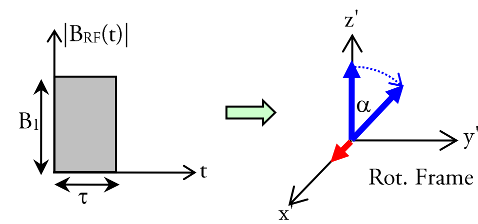
The magnetization **M** precesses about the x axis in the rotating frame. We can thus create any angle we'd like between it and the z-axis, depending on how long we let it precess and how strong B_1 is. Let's assume we have B_{RF} on for just enough time for the magnetization to tilt to the xy plane - that is, create a 90° angle between B_0 and **M**. Deducing the motion of **M** in the lab frame is now merely a matter of transforming back to the lab frame, which simply rotates at an angular velocity $-\omega_{rot}$ relative to the rotating frame. That is, **M** in the lab frame performs a spiral as it descends and rotates:



Shown here is the trajectory of the magnetization **M** in the lab (left) and rotating (right) frames. The two frames are connected by a simple rotation.

Setting The Radiofrequency (RF) Pulse's (Area)=(Duration)×(Amplitude) Sets The Flip Angle on Resonance

We see the spins will perform a rotation about the x-axis in the rotating frame at a frequency $\omega_1=\gamma B_1$. Note this is **not the same as** ω_{RF}, (one is the amplitude of B_{RF}, the second is its oscillating frequency). After a time τ, **M** will have created an angle $\alpha = \omega_1 \tau = \gamma B_1 \tau$:



Note that

$$\alpha = \gamma(\text{amplitude of RF}) \times (\text{duration of RF}).$$

This relation is true only on resonance, when $\omega_{RF}=\omega_0$, where $\mathbf{B}_{eff}$ has no z-component.

To "tip" the magnetization onto the y axis, we wait a time t_{90} such that:

$$\alpha = \gamma B_1 t_{90} = \tfrac{\pi}{2},$$

or

$$t_{90} = \frac{1}{4\gamma B_1}.$$

In the original laboratory (unrotating) frame the spins execute additional motions, but the important thing to realize is that **a spin which is in the xy plane in the rotating frame, must also be in the xy-plane in the laboratory frame** (although *where* in the plane is a different story).

> **Number Time**. We've remarked that $B_{1,max}$ ~ 10 μT for an MRI scanner. For protons, one would need $t_{90} = \frac{\pi}{2\gamma B_1}$ ~ 0.5 ms to excite the spins onto the xy-plane. For ^{13}C, $t_{90} = \frac{\pi}{2\gamma B_1}$ ~ 2 ms .

Relaxation can be Neglected During Excitation Since Most Pulses are Shorter Than T_1, T_2

Our calculations in the previous section have shown that excitation mostly happens on the timescale of milliseconds in MRI, which is much shorter than T_1, T_2. Hence, to an excellent approximation, relaxation effects can be neglected for most pulses and most tissue types in the body. We will make some remarks about the effects of relaxation later on but, in general, will neglect it unless specifically stated otherwise.

The Phase of the Pulse Determines the Phase of the Excited Magnetization

We have so far modeled B_{RF} in the lab frame as:

$$\mathbf{B}^{(RF,lab)} = \begin{pmatrix} B_1\cos(-\omega_c t) \\ B_1\sin(-\omega_c t) \\ 0 \end{pmatrix}.$$

Since we have full control over the x and y component we have no problem modulating both $B_1(t)$ and adding a time-dependent phase $\phi(t)$ to the RF field:

$$\mathbf{B}^{(RF,lab)} = \begin{pmatrix} B_1(t)\cos(-\omega_c t + \phi(t)) \\ B_1(t)\sin(-\omega_c t + \phi(t)) \\ 0 \end{pmatrix}$$

In the rotating frame, this will look like this[1]:

$$\mathbf{B}^{(RF,rot)} = \begin{pmatrix} B_1(t)\cos(\phi(t)) \\ B_1(t)\sin(\phi(t)) \\ 0 \end{pmatrix}.$$

Let's keep $B_1(t)$ and $\phi(t)$ fixed. Then the constant phase $\phi(t)=\phi_0$ is called the **phase of the pulse**, and is equal to the angle the RF field makes with the x-axis. determines where the RF pulse will point in the transverse plane.

The **phase of the magnetization** is defined as the angle made by the transverse component of the magnetization vector (i.e. its projection on the xy plane) with the x-axis.

Because the magnetization gets tipped at right angles to the RF field following the left hand rule, the relation between the pulse's and magnetization's phase ϕ_m is given by:

$$\phi_m = \phi_{RF} + \frac{\pi}{2}.$$

The standard notation for a constant RF pulse then assumes the form α_ϕ, where α is its flip angle and ϕ its (constant) phase. The following conventions are also used:

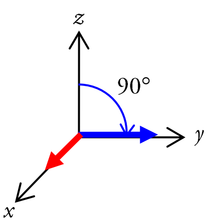
$$"x": \phi_{RF} = 0°$$
$$"y": \phi_{RF} = 90°$$
$$"-x": \phi_{RF} = 180°$$

[1] To prove this, use $\mathbf{B}^{(rot)} = R_z(\omega_c t)\mathbf{B}^{(lab)}$, where $R_z(\omega_c t)$ is a RH rotation matrix about the z-axis by an angle $\theta = \omega_c t$ (the rotating frame rotates with a left handed rotation and angular frequency ω_c; in it, it appears the RF field rotates at the same angular frequency but in the opposite direction). There is a bit of algebra and trigonometry involved but the proof is straightforward.

$$"-y": \phi_{RF} = 270°$$

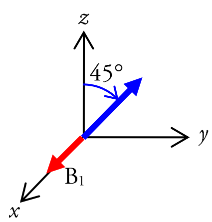
Some examples are shown below (magnetization is assumed to start out from z, and is the blue vector; the RF is the red vector):

(90°)$_x$ pulse



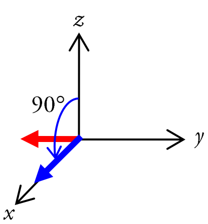
Magnetization in yz plane (phase: 90°)

(45°)$_x$ pulse



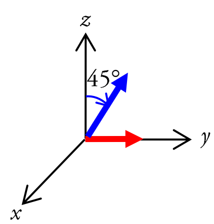
Magnetization in yz plane (phase: 90°)

(90°)$_{-y}$ pulse



Magnetization in xz plane (phase: 0°)

(45°)$_y$ pulse



Magnetization in xz plane (phase: 180°)

Flip Angles < 90° Minimize Duration & Decrease Power Deposition at the Cost of SNR

An excitation pulse need not tip the spins by 90°, and can create any angle α between **M** and the main $\mathbf{B_0}$ field. The disadvantage of this is its reduced signal: in our simple model we've seen that the voltage,

$$\varepsilon = -\frac{\mu_0}{2R}\frac{dM_y}{dt},$$

is proportional to the time derivative of M_y (reorienting the coil would introduce the time derivative of M_x, and would not change our conclusions). The magnitude of M_y will be proportional to the flip angle. Hence, the signal itself will also be proportional to sin(α) and decrease with the flip angle[2]:

$$\text{signal} \propto M_y, M_x \propto \sin(\alpha).$$

On the other hand, the pulse's duration,

$$t_\alpha = \frac{\alpha}{\gamma B_1},$$

is proportional to the flip angle and decreases linearly (assuming we keep B_1 fixed).

Another advantage of short pulses is that they have reduced specific absorption rate (SAR). Some of the RF energy is absorbed in the patient's tissue and causes undesired heating. The amount of SAR is proportional to the square of B_1 and the pulse's duration:

$$\text{SAR} \propto \int_0^t \left|B_1(t)\right|^2 dt \overset{\text{constant pulse, duration } t_\alpha}{=} B_1^2 t_\alpha = \frac{\alpha B_1}{\gamma}.$$

We observe SAR reduces linearly with the flip angle. The amount of SAR is limited by most modern scanners' hardware based on our understanding of the effect of SAR on biological tissues. Modern RF coils deposit power on par with modern cell-phones and are generally considered safe as long as guidelines are observed.

Off-Resonant Excitation: The Concepts of Bandwidth & Selective Excitation

The z-Field can vary as a Function of Position, Which Leads to Non-zero Offsets in the Rotating Frame

So far our approach has been to make B_0 disappear by moving to a rotating frame at a

[2] We will see later on this decrease is actually mitigated in most sequences where pulses are applied rapidly (on the order of, or faster than T_1) and don't afford the magnetization enough time to return to thermal equilibrium before the next excitation.

frequency $\omega_{rot} = \omega_0 = \gamma B_0$, in which the fictitious field negates B$_0$ completely:

$$B_z^{(lab)} = B_0 \rightarrow B_z^{(eff)} = B_0 - \tfrac{\omega_{rot}}{\gamma} = 0 .$$

However, when B$_0$ varies as a function of position, $B_0 = B_0(\mathbf{r})$, it is impossible to make the z-component of the field disappear at every point:

$$B_z^{(lab)} = B_0(\mathbf{r}) \rightarrow B_z^{(eff)} = B_0(\mathbf{r}) - \tfrac{\omega_{rot}}{\gamma} \neq 0 .$$

Some sources of variation could include:

1. Imperfections in the main magnet field.
2. Susceptibility artifacts in the sample, in which the external field induces microscopic magnetic moments which themselves distort the main field (in all directions, but predominantly in the direction on B$_0$).
3. Some patients might have metal implants which distort the magnetic field – again, in many directions, but their effect is most pronounced along B$_0$.
4. Often we <u>intentionally</u> create these inhomogeneities, as is the case with gradient coils, in which we create a linear dependence of the z-field on position:

$$B_0 \rightarrow B_0 + \mathbf{G}(t) \cdot \mathbf{r} .$$

For example, when a gradient is turned on,

$$B_z^{(lab)} = B_0 + \mathbf{G}(t) \cdot \mathbf{r} \rightarrow B_z^{(eff)} = \mathbf{G}(t) \cdot \mathbf{r} ,$$

and, when the RF is turned on:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \mathbf{G}(t) \cdot \mathbf{r} \end{pmatrix}$$

If we have some form of spatial inhomogeneity due to hardware imperfections/susceptibility artifacts, we could write it as

$$B_z^{(lab)} = B_0 + \Delta B(\mathbf{r}) ,$$

and in the rotating frame its z-component will be

$$B_z^{(eff)} = B_z^{(lab)} - \tfrac{\omega_{rot}}{\gamma} = \Delta B(\mathbf{r}) .$$

These effects are all cumulative: If we have both imperfections in B$_0$ **and** a gradient turned on, the effective field will be:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \mathbf{G}(t) \cdot \mathbf{r} + \Delta B(\mathbf{r}) \end{pmatrix} .$$

<u>Number Time</u>. A gradient will create a range of frequencies given by $\gamma\!\!\!/ G \Delta z$ over a spatial region of width Δz. Across a 1 mm pixel, this will be $\gamma\!\!\!/ G \Delta z \approx 420$ Hz for G=10 mT/m. Susceptibility artifacts at 3 Tesla will create spatial variations across the head on the order of hundreds of Hz, mostly in regions where air-tissue interfaces exist such as the prefrontal cortex, close to the oral cavity or ears, and so forth.

In the case the z-component is not completely zeroed out, we must analyze and understand the case for which

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} .$$

The quantity ΔB will be referred to as the **offset** of the spins. Since we will be looking at a specific point in space we can assume ΔB is just a constant.

All Pulses Are Selective With a Finite Bandwidth (BW) Given by ~ $1/\gamma B_1$

It is fairly simple to divide our analysis into two extreme cases: in one, $\Delta B \ll B_1$, and we can neglect it, obtaining:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} \approx \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}$$

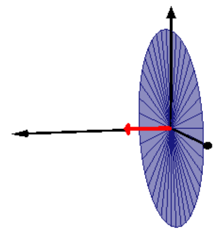
We thus recover the previous case in which we excite the spins "as usual", as if they were on resonance. On the other extreme, $\Delta B >> B_1$,

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ \Delta B \end{pmatrix}.$$

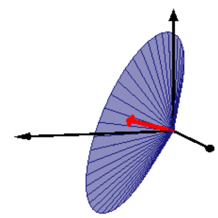
and the RF excitation will have no effect, resulting in no excitation. We can guess and extrapolate between these two extremes, saying that there is a cutoff to the effect of B_1 when $B_1 \sim \Delta B$. In other words, a range of offsets $\Delta B \sim B_1$ will be excited. This is known as the **bandwidth** of the pulse: the range of offsets (or frequencies) it will excite.

> The **bandwidth** (BW) of a pulse is given by the range of frequencies it affects. For the excitation pulse we've just discussed,
>
> $$BW = \gamma B_1 .$$

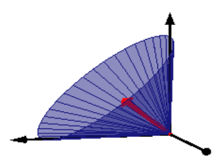
This can also be understood graphically, by plotting the precession cone of the spins about the effective field, starting out from thermal equilibrium (i.e. **M** along the z-axis):



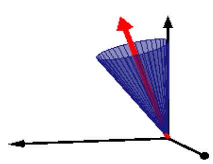
$\Delta B=0$
B_{eff} points in the xy-plane – say, along the x-axis in this example - and the spin precesses in a circle in the yz plane.



$\Delta B \ll B_1$
B_{eff} starts tilting up in the xz plane, causing the rotation cone to start "folding".



$\Delta B=B_1$
In this "dividing case", the x- and z- components of B_{eff} are equal. The "precession cone" just touches the xy plane.



$\Delta B \gg B_1$
B_{eff} now becomes close to the z-axis. The precession cone becomes very narrow: even if we wait for a long amount of time the spins will not stray far from the z-axis.

A Hard Pulse is One With High Peak Power And An "Infinite" Bandwidth

When we are interested in flipping all of the spins onto the xy-plane regardless of their offset we must create a bandwidth larger than the range of offsets in our sample. Since $BW = \gamma B_1$, this means we need to have a very high B_1>>range of offsets in our sample. The duration of an (on-resonance) 90°-pulse,

$$t_{90} = \frac{1}{4\gamma B_1} .$$

We see that hard pulses are short and have a high peak power. Such pulses are called **hard pulses** in MR jargon. The "ideal" hard pulse is one for which $B_1 \rightarrow \infty$, duration→0, such that $\gamma B_1 \cdot (duration)$= the desired flip angle.

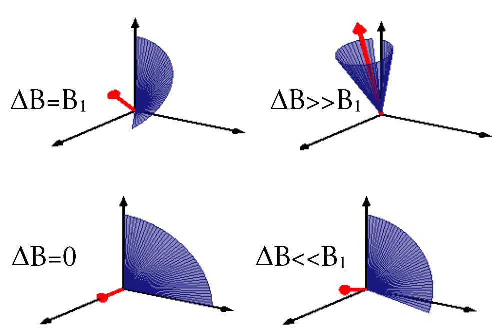
A Constant Pulse's Excitation Profile

Let us explore what happens when our pulse is not "hard" and has a finite duration. We'll look at a 90° pulse, although our conclusions will apply to any pulse flip angle.

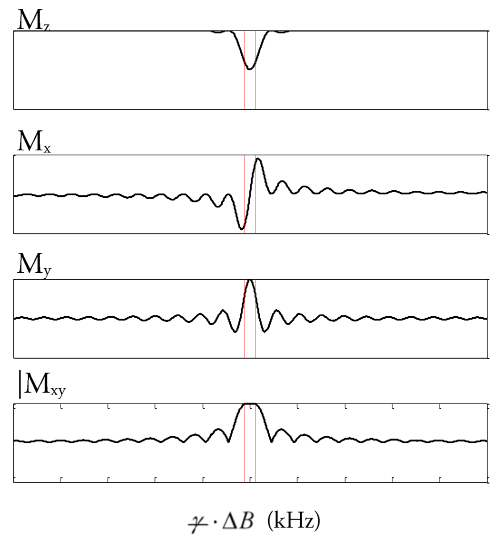
For a 90° flip angle, the duration must be

$$t_{90} = \frac{1}{4\bar{\gamma}B_1}.$$

However, as previously discussed, this only ensures a 90° flip angle for spins when the offset is zero: ΔB=0. As we increase ΔB, the magnitude and direction of $\mathbf{B}_{eff}$ and its direction vary, and the corresponding final position of the magnetization - assumed to start out from thermal equilibrium along the z-axis - varies. In fact, even if B_1 is applied along x, spins not at the center do not even remain in the yz plane anymore. The following diagram shows the effective field and the precise trajectory traced by the magnetization vector during the pulse's duration, t_{90}, for the four cases outlined previously:



Instead of these pictorial diagrams, one can plot the components of **M** as a function of the offset, ΔB. This is known as the pulse's **frequency response** or **pulse profile**. Such a response is plotted below for $\bar{\gamma}B_1$ = 1 kHz (B_1≈23.5 μT), t_{90}=0.25 ms (B_1 applied along the x-axis as in the above diagrams, i.e. has 0° phase):



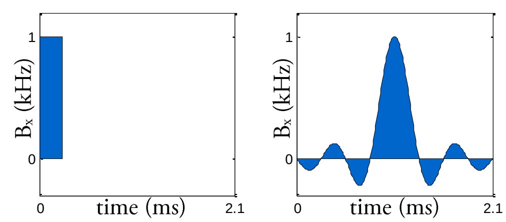
In the above, $|M_{xy}| = \sqrt{M_x^2 + M_y^2}$. The magnetization was assumed to have an arbitrary magnitude of unity, and the vertical axis stretches from ±1. The dashed red lines signify the points at which $\bar{\gamma}\Delta B = \bar{\gamma}B_1 = 1\ kHz$, which define the bandwidth of the pulse. It is quite clear that the concept of bandwidth has some artbitrariness to it since the profile of M_z and M_{xy} are not sharp.

The profile of M_{xy} actually looks somewhat wider than M_z, which is a result of the magnetization vector's constant magnitude, $|\mathbf{M}|^2 = |M_x|^2 + |M_y|^2 + |M_z|^2 = |M_{xy}|^2 + |M_z|^2 = 1$ and the relation between M_z and $|M_{xy}| = \sqrt{1-|M_z|^2}$. For example, if $|M_z|$=0.9, then $|M_z|^2 = 0.81$ and $|M_{xy}| = 0.436$. So, even if $|M_z|$ is almost unperturbed, $|M_{xy}|$ might still appear to be quite sizable, leading to its wider profile.

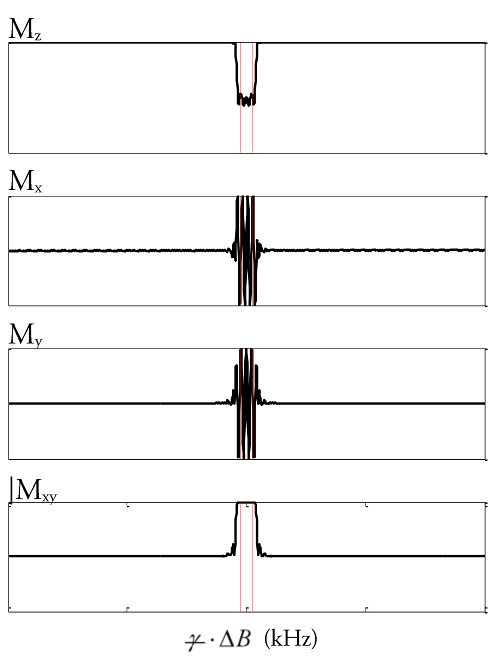
Also note the extensive "wiggles" in M_{xy} outside the slice, indicating that some excitation occurs even for ΔB>>B_1. We will deal with this in a moment by introducing shaped pulses.

Shaping the Pulse Affects the Pulse's Profile

Modern RF transmitters have the capability of shaping the RF pulse, $B_{RF}(t) = B_x(t) + iB_y(t)$; that is, controlling its x- and y- components. Such pulses are called **shaped pulses**. For the constant pulse we had $B_x=B_1$, $B_y=0$. Let us see what happens if we vary $B_x(t)$ in a sinc-like manner:



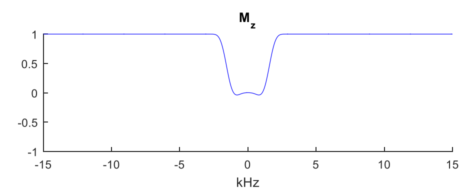
The new pulse maintains the same peak B_1 and same area (and hence flip angle) as the rectangular pulse, but is necessarily longer (since the negative lobes of the sinc detract from the area). The frequency response of this pulse can be calculated by solving the Bloch equations numerically, yielding:



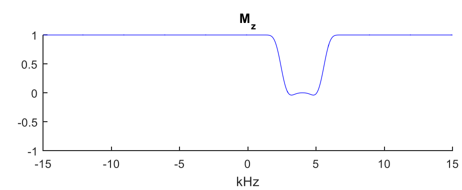
The response is shown using the same scaling and plot range as the rectangular pulse for a "fair" comparison. The dashed red lines represent the same bandwidth $(= \bar{\gamma} B_1)$ calculated for the rectangular pulse. The ensuing response is significantly better-behaved, with less wiggles and sharper transition lines. We won't go into the theory of shaped pulses in this course, but we will remark without proof that for tip angles up until about 90° the profile of M_{xy} resembles the Fourier transform of $B_{RF}(t)$.

The Slice's Center Frequency can be Shifted by Sinusoidally Modulating the Pulse's Phase

The pulses discussed so far excite a bandwidth about a central frequency $\bar{\gamma}\Delta B = 0$:



However, one is often interested in simply exciting a different region of frequencies, i.e. to shift the response's center:



There is a very simple way to shift the center of the excited slice, by modulating the pulse's phase with a linear term. Mathematically, this means we're **adding a phase that varies linearly with time** to the pulse:

$$\boldsymbol{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta\omega/\gamma \end{pmatrix} \rightarrow \begin{pmatrix} B_1 \cos(\omega_c t) \\ -B_1 \sin(\omega_c t) \\ \Delta\omega/\gamma \end{pmatrix} \equiv \boldsymbol{B}_{eff}^{(shift)}$$

To understand why this shifts the pulse, imagine being given $\mathbf{B}_{\text{eff}}^{(\text{shift})}$ in the rotating frame, where it rotates around the z-axis according to the left hand rule with angular frequency ω_c. By performing a *second* rotating frame transformation, into a frame which rotates with ω_c relative to the original ("first") rotating frame, we fix B_1 and add an additional fictitious field:

$$\boldsymbol{B}_{eff}^{(shift)} \xrightarrow[frame]{2^{nd}\ rotation} \begin{pmatrix} B_1 \\ 0 \\ \dfrac{\Delta\omega}{\gamma} - \dfrac{\omega_c}{\gamma} \end{pmatrix}$$

In this 2nd rotating frame the z-component has a fixed offset. All of our previous arguments can be repeated: the center frequency of the excitation profile will occur on-resonance, when the z-component is zero: $\Delta\omega - \omega_c = 0$, or:

$$\Delta\omega = \omega_c.$$

Assuming $\omega_{rot} = \omega_0$, then $\Delta\omega = \gamma G z$. The position at which the spins will be on resonance ($\Delta\omega = \omega_c$) above condition will occur will be determined by $\gamma G z_c = \omega_c$, or

$$z_c = \frac{\omega_c}{\gamma G}$$

For example, if we have a G=1 mT/m gradient on, and we want to shift the slice by 1 cm (z_c=0.1 m), we need to set ω_c such that

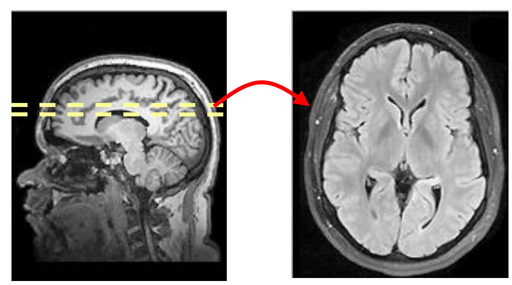
$$\omega_c = \gamma G z_c \approx 2\pi \cdot 4.257\ kHz$$

Current MRI hardware enables one to control the phase and amplitude of the RF pulse as a function of time, making shifting the profile's center easy (we can generate any practical frequency ω_c). It also places almost no demands on the hardware[3].

Slice Selection

Often, We Are Interested in Exciting a Single Slice

We now come to the first form of spatial selectivity in MRI: **selective excitation**, in which only a part of the sample is excited. We will confine ourselves to the simple scenario of one-dimensional excitation, meaning selectively exciting a range of positions along a fixed axis:



Without such selective excitation, an RF pulse would excite the entire sample. Although this can be and is sometimes done, a slice-selective approach also has its own merits.

Applying a Pulse in the Presence of a Gradient Will Excite a Slice

In the joint presence of a gradient and an RF irradiation, the effective field in the rotating frame is:

$$\boldsymbol{B}_{eff}(\boldsymbol{r}, t) = \begin{pmatrix} B_1 \\ 0 \\ \boldsymbol{G}(t) \cdot \boldsymbol{r} \end{pmatrix}$$

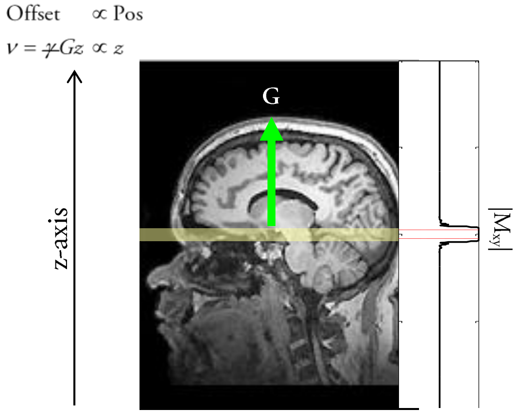
.

[3] This "trick" for shifting the pulse's profile also holds for non-constant pulses.

We will assume for simplicity our gradient is constant and turned on along the z-direction, so $\mathbf{G}(t)=G\hat{\mathbf{z}}$, and so:

$$\boldsymbol{B}_{eff}(\boldsymbol{r},t)=\begin{pmatrix}B_1\\0\\Gz\end{pmatrix}$$

The gradient creates a linearly increasing offset along z:



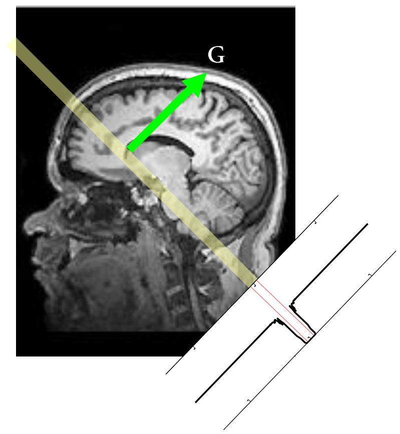
The gradient assigns frequencies to positions via $\nu=\gamma Gz$, and hence any pulse that excites a range of frequencies $BW=\gamma B_1$ will, in the presence of a gradient, excite a range of positions given by:

$$\Delta z=\frac{BW}{\gamma G}=\frac{B_1}{G}$$

(for a fixed flip angle)

This is, in fact, the **slice thickness**.

The excited slice will be perpendicular to the direction of the gradient vector **G**. For example applying the same pulse with a gradient in a different direction - say, at 45° to the z-axis - will excite a slice that is itself tilted by 45°, since now our gradient will create a linear correspondence between frequency and the x+z=const planes:



Finally, just as the profile of a pulse can be shifted in frequency space as a function of the offset, thus the slice's center can also be shifted by simply modulating the RF pulse's shape with a linear phase.

> SUMMARY: HOW TO SET A SLICE'S POSITION
> 1. A slice's width can be adjusted by varying either the pulse's bandwidth (via its amplitude, B_1) or the gradient's amplitude, G=|**G**|.
> 2. The slice's orientation can be adjusted by adjusting the gradient's orientation, $\hat{\mathbf{G}}$.
> 3. The slice's position along the gradient's axis can be adjusted by adding a linear phase (in time) to the pulse.

Setting the Gradient too Low Might Distort the Slice Profile

Note there are two ways of controlling a slice's width, via the pulse or the gradient. There is some freedom in choosing which one to vary; however, several things limit this freedom in practice:

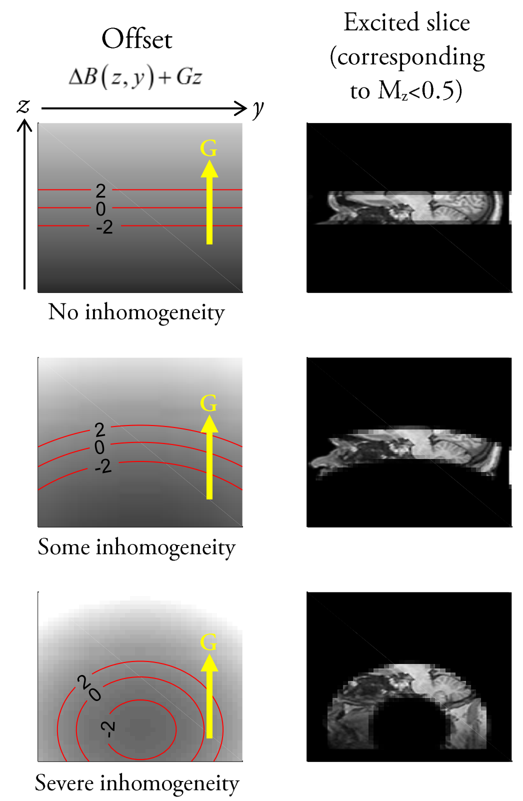
1. The maximal magnitude of G is limited by the hardware. For a typical human MRI scanner, this is on the order of ~10-100 mT/m.

2. The maximal B_1 (and therefore maximal bandwidth) is also limited.
3. For most pulses, one cannot simply change B_1 or the pulse's bandwidth without changing the pulse. For example, if one wants to keep the flip angle fixed ($\alpha = \gamma B_1 T$ for a constant pulse), the pulse needs to be made shorter or longer. This is often undesirable since the timing of events has crucial meaning in many excitation schemes. Thus, in practice, whenever possible G is changed before B_1 is changed.

On the other hand, making G too small would make the range of frequencies across the slice small as well. This might interfere with the slice profile and distort it. Shown below is the excited region in a 2D image in the *yz*-plane for three levels of inhomogeneity, for a simple excitation pulse with a bandwidth of 4 kHz. The offset as a function of position (= $\Delta B(z,y)+Gz$) is shown on the left, scaled to between ±15 kHz.

In the homogeneous case (top), all frequencies between ±2 kHz are excited. These limits are clearly drawn using contour lines. In the second example, some quadratic inhomogeneity is added, $\Delta B(\mathbf{r}) = \alpha\left(z^2+y^2\right)$, with α=1 Hz/mm^2 and z, y given in mm. This creates a spatially varying inhomogeneity, which reaches 5 kHz at the edges of the image, on the order of the pulse's bandwidth. As a result, the ±2 kHz contour lines of the offset $\Delta B(z,y)+Gz$ get distorted, as does the slice's profile. In reality, this would be a pretty bad case of inhomogeneity – one does not encounter this magnitude of effects over the entire head, although some local regions (such as the prefrontal cortex close to the sinuses) can exhibit some large local inhomogeneities. This would cause the slice profile to be distorted close to the sinuses.

The bottom case represents a case of severe inhomogeneity, with α=3 kHz/mm^2 and a maximal inhomogeneity of 15 kHz at the image's edges. The resulting contour lines are even more distorted, and the "slice" now becomes a circle! These effects are extreme and not encountered in reality.



In practice, one is given a pulse, and then the only way to control the slice's width is by controlling G. One must then be careful not to set G too small so as to generate spatial distortions (or at the very least be aware of them and account for them during post-processing somehow).

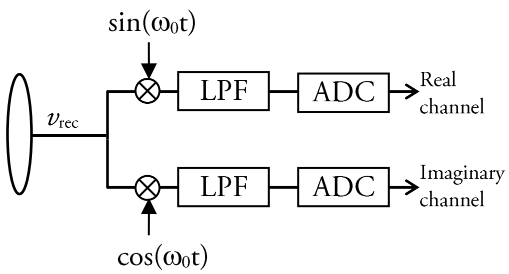
Signal Acquisition

The Acquired Signal in MRI is Proportional to the Transverse Magnetization

The expression for the signal we've derived previously depends on the magnetization in the lab frame:

$$v_{rec} = -\int_{body} \widehat{\boldsymbol{B}}_{rec}(\boldsymbol{r}) \cdot \frac{d\boldsymbol{M}(\boldsymbol{r},t)}{dt} dV$$

This expression relies on the magnetization in the lab frame. It is possible to relate this to the magnetization in the rotating frame, given the particular way signals are processed and digitized:



This is done in the appendix and is quite messy. Interested readers are referred to the appendix. We here simply state the result:

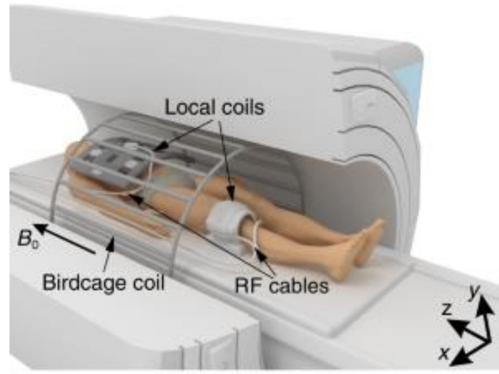
$$s(t) \propto \omega_0 \int_{body} \overline{B_{xy}^{(rec)}(\boldsymbol{r})} M_{xy}^{rot}(\boldsymbol{r},t) dV$$

where

$$B_{xy}^{(rec)}(\boldsymbol{r}) = B_x^{(rec)}(\boldsymbol{r}) + iB_y^{(rec)}(\boldsymbol{r})$$

is a complex number, with its real part equal to the x-component of the receiver field and its imaginary component equal to the y-component of the receiver field[4].

The MRI scanner has a large body coil that is used for reception, shaped like a "birdcage". It creates a uniform, static field in the (xy) plane:



In this case, $B_{xy}^{(rec)}$ is approximately spatially uniform throughout the body, and one can approximate:

$$s(t) \propto \omega_0 \int_{body} M_{xy}^{rot}(\boldsymbol{r},t) dV$$

Often, reception is done with a coil that picks up signal only from a limited region in space (the local coils in the picture above, designed to acquire signal from the chest and the knee). In that case, B_{xy} will be localized to the region being scanned. We will often make the approximation to 0^{th} order that the field is constant. For example, for the knee coil:

$$s(t) \propto \omega_0 \int_{knee} M_{xy}^{rot}(\boldsymbol{r},t) dV$$

We will make use of the spatially varying profile of the coils when we come to talk about parallel imaging, but until then, we'll keep the "spatially uniform acquisition" assumption throughout the upcoming lectures.

[4] This is the field generated by the receiver coil when you put unit current (1 Amp) through it.

Appendix

Proof of the Signal Acquisition Expression

Armed with our expression for the time course of the magnetization in lieu of RF pulses,

we turn to deriving a usable expression for the acquired signal (voltage) in our receiver coil. We've seen in Lecture 3 that, for any receiver coil,

$$\begin{aligned} v_{rec} &= -\int_{\text{body}} \mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d\mathbf{M}(\mathbf{r},t)}{dt} dV \\ &= -\int_{\text{body}} \left[B_x^{(rec)} \frac{dM_x}{dt} + B_y^{(rec)} \frac{dM_y}{dt} + B_z^{(rec)} \frac{dM_z}{dt} \right] dV \end{aligned}$$

.

Here **M** and $\mathbf{B}_{rec}$ are both measured in the laboratory frame. First, the time derivative of the z-component of the magnetization, which changes with a time constant T_1 (on the order of Hz) is much smaller than the x- and y-components, which precess with a frequency of MHz. Therefore, we can neglect the z-component to an excellent approximation:

$$v_{rec} \approx -\int_{\text{body}} \left(B_x^{(rec)} \frac{dM_x}{dt} + B_y^{(rec)} \frac{dM_y}{dt} \right) dV .$$

(Assuming T_1 is slow compared to ω_0^{-1})

We define $M_{xy} = M_x + iM_y$, $B_{xy} = B_x + iB_y$ and note:

$$\begin{aligned} \overline{B_{xy}} M_{xy} &= (B_x - iB_y)(M_x + iM_y) \\ &= B_x M_x + B_y M_y + i(B_x M_y - M_x B_y) \end{aligned}$$

so

$$B_x M_x + B_y M_y = \text{Re}\left(\overline{B_{xy}} M_{xy}\right).$$

Therefore:

$$v_{rec} = -\text{Re}\int_{\text{body}} \overline{B_{xy}^{(\text{rec})}} \frac{dM_{xy}^{(\text{lab})}}{dt} dV .$$

Since we have an expression for M_{xy} in the lab frame, we can differentiate it. Since ω_0 varies much faster than T_2 and $\gamma\mathbf{G}(t)\cdot\mathbf{r}$, we can neglect both terms and obtain[5]:

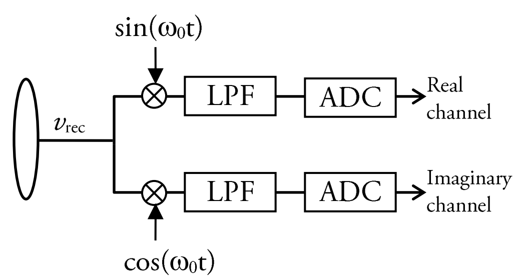
$$\frac{dM_{xy}^{(\text{lab})}(\mathbf{r},t)}{dt} \approx -i\omega_0 M_{xy}^{(\text{lab})}(\mathbf{r},t),$$

(Assuming ω_0 is faster than any offsets)

so

$$v_{rec} = \omega_0 \,\text{Re}\int_{\text{body}} i\overline{B_{xy}^{(\text{rec})}(\mathbf{r})} M_{xy}^{(\text{lab})}(\mathbf{r},t) dV$$

The interesting signal is modulated by a rapid (~100 MHz) phase term, $e^{-i\omega_0 t}$, which is "hidden" inside M_{xy}. It is beneficial to get rid of it for both convenience, as well as to lessen the burden on the analog-to-digital converter (which needs to deal with slower varying signals). This is called **demodulation**. To do this, v_{rec} is split into the identical copies, with one multiplied by $\cos(\omega_0 t)$ and the other by $\sin(\omega_0 t)$, and each is then passed through a low-pass filter (LPF)



Using

[5] This assumption needs to be modified when studying solids with very short T_2s (on the order of microseconds). However, such short T_2s are rarely observed in MRI since they lead to signals well below the noise levels.

$$\cos(-\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$
$$\sin(-\omega_0 t) = \frac{e^{-i\omega_0 t} - e^{i\omega_0 t}}{2i}$$

and writing

$$M_{xy}^{(\mathrm{lab})}(\mathbf{r},t) = \left|M_{xy}^{(\mathrm{lab})}(\mathbf{r},t)\right| e^{i\phi_M(\mathbf{r},t)}$$
$$B_{xy}^{(rec)}(\mathbf{r},t) = \left|B_{xy}^{(rec)}(\mathbf{r},t)\right| e^{i\phi_B(\mathbf{r},t)}$$

we obtain, right before the LPF:

$$(v_{rec})_{im} = \tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} i\left|B_{xy}\right|\left|M_{xy}\right| \times\left[e^{i[\omega_0 t+\phi_M-\phi_B]} + e^{i[-\omega_0 t+\phi_M-\phi_B]}\right]dV$$
$$(v_{rec})_{re} = \tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} \left|B_{xy}\right|\left|M_{xy}\right| \times\left[e^{i[-\omega_0 t+\phi_M-\phi_B]} - e^{i[\omega_0 t+\phi_M-\phi_B]}\right]dV$$

Since the LPF removes the fast changing component $-\omega_0 t + \phi_M - \phi_B$, we obtain, after the LPFs:

$$(v_{rec})_{im} = \tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} i\left|B_{xy}\right|\left|M_{xy}\right| e^{i[\omega_0 t+\phi_M-\phi_B]}dV = -\tfrac{\omega_0}{2}\int_{\mathrm{body}}\left|B_{xy}\right|\left|M_{xy}\right|\sin(\omega_0 t+\phi_M-\phi_B)\,dV$$
$$(v_{rec})_{re} = -\tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} \left|B_{xy}\right|\left|M_{xy}\right| e^{i[\omega_0 t+\phi_M-\phi_B]}dV = -\tfrac{\omega_0}{2}\int_{\mathrm{body}}\left|B_{xy}\right|\left|M_{xy}\right|\cos(\omega_0 t+\phi_M-\phi_B)\,dV$$

We then form the complex signal in the computer:

$$s(t) = (v_{rec})_{re} + i(v_{rec})_{im} = -\tfrac{\omega_0}{2}\int_{\mathrm{body}}\left|B_{xy}^{(rec)}\right|\left|M_{xy}^{(\mathrm{lab})}\right| e^{i(\omega_0 t+\phi_M-\phi_B)}dV$$

Since $\left|M_{xy}^{(\mathrm{lab})}\right| = \left|M_{xy}^{(\mathrm{rot})}\right|$, and since the rotating and lab frame magnetization vectors are related via $M_{xy}^{(\mathrm{rot})}(\mathbf{r},t) = M_{xy}^{(\mathrm{lab})}(\mathbf{r},t)e^{i\omega_0 t}$, we can simplify:

$$s(t) \propto \omega_0 \int_{\mathrm{body}} \overline{B_{xy}^{(rec)}(\mathbf{r})}\, M_{xy}^{(\mathrm{rot})}(\mathbf{r},t)\,dV$$

We've omitted the constant of proportionality since the actual measured signal's magnitude will depend anyway on the electronics, amplifiers and so on.