LECTURE 4

EXCITATION & ACQUISITION

Lecture Notes by Assaf Tal

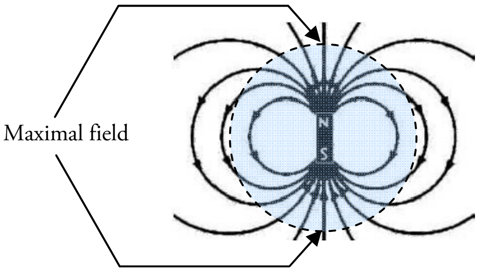
TO MEASURE A SIGNAL, SPINS MUST BE EXCITED

The Static Nuclear Magnetic Field Is Too Weak To Be Reliably Detected

We've previously calculated the bulk magnetic moment of 1 mm^3 of water in a 3 T magnetic field and found it to be about $|\mathbf{M}| \sim 10^{-8}$ J/T. Imagine this voxel is inside the human body and must be detected in a coil wrapped around the body – say, 20 cm away from it. The magnetic field of the voxel is dipolar

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{M}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{M}}{r^3}$$

If we consider a sphere of radius r around **M**, the field will be maximal at the poles:



where its value will be (putting r=0.2 m, M=10^{-8} J/T, $\mu_0=4\pi\cdot 10^{-7}$ N/A^2):

$$|\mathbf{B}(\mathbf{r})| = \frac{\mu_0 M}{2\pi r^3} \sim 10^{-13} \text{ T=0.1 pT}.$$

This is an incredibly small magnetic field. The extremely weak magnetic fields produced by neural currents are 1-2 orders of magnitude larger, and even those require extremely specialized hardware based on super-cooled magnetic field detectors known as SQUIDs. These are used in magnetoencephalography (MEG) machines, but they are not sensitive enough to detect the sort of signals we're interested in. It is possible to construct sensitive enough detectors, but even if we put aside the engineering complexity and cost of these in addition to the MRI magnet itself, we are still faced with furhter problems:

- How can we separate the tiny field created by the nuclear spins from the much larger sources of magnetic fields created by other phenomena in the body, e.g. neural currents?
- And how can we, by detecting the magnetic fields outside the body, deduce the distribution of spins within the body? That is, can we **image** $\mathbf{M}(\mathbf{r})$ by measuring $\mathbf{B}(\mathbf{r})$ outside the body? These sort of problems are known as **inverse problems** and are often incredibly difficult to solve properly (the same issue plagues MEG).

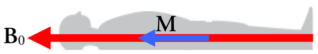
Is there a better way to detect the MRI signal? The answer is yes, and it is linked to the precession of spins around the main B_0 field.

Precessing Spins Induce Measurable Currents In The Receiver Coils

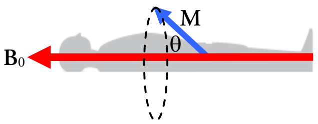
The **R** in **MRI** stands for **resonance**. As we've remarked previously, any spin will precess about a constant magnetic field. In particular, if we place the spins in a static, strong magnetic field – say, the 3 Tesla field of a typical MRI scanner – it will precess. For a hydrogen nucleus, this precession frequency would be

$$\nu = \gamma\!\!\!{}^{-} B_0 = 42.57\,\tfrac{\text{kHz}}{\text{mT}}\cdot 3\text{ T} \approx 127\text{ MHz}.$$

Using a coil and the law of reciprocity we can measure the time-depdendent flux induced by the spin. This dynamic measurement will create a significantly larger signal than a static one. We are, however, faced with a paradox: at thermal equilibrium, the bulk magnetic moment is parallel to B_0, and hence the precession is "degenerate" – **M** remains static (even though the microscopic moments, **m**, will precess). For us to observe true precession, **M** must make some non-zero angle with B_0:

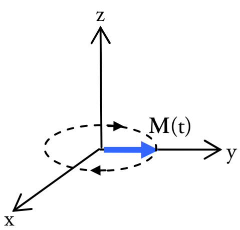


At thermal equilibrium, **M** and $\mathbf{B}_0$ are colinear, and no precession is observed.



Only by creating some nonzero angle θ between **M** and $\mathbf{B}_0$ – that is, only by **exciting M** - can we observe precession.

Creating such an angle is called **excitation**. Putting that aside for a moment, let's calculate the voltage induced in a coil put around the precessing spin via Faraday's law and the law of reciprocity. Here we take a wo circular coils of radius R in the xz and yz planes, with a point magnetic moment placed at the origin and performing some rotation as a function of time:



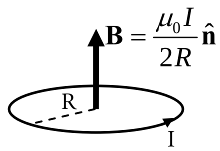
The components of the precessing spin have a sinusoidal time dependence:

$$\mathbf{M}(t) = \begin{pmatrix} M\cos(\omega t) \\ -M\sin(\omega t) \\ 0 \end{pmatrix} \qquad \text{LH rotation about z .}$$

Remember that the law of reciprocity tells us that the voltage induced in the coil can be calculated via

$$v_{rec} = -\mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d\mathbf{m}}{dt},$$

where $\mathbf{B}_{rec}$ is the field created by a unit current the loop at the position of the magnetic moment (at the origin). The expression for the magnetic field created by a loop of current at its center is well known from basic magnetism:



where R is the ring's radius, I the current, and $\hat{\mathbf{n}}$ a unit vector normal to the plane of the ring. We take unit current (I=1) and so:

$$\varepsilon = -\frac{d}{dt}\left(\frac{\mu_0}{2R}\hat{\mathbf{y}} \cdot \mathbf{M}\right) = -\frac{\mu_0}{2R}\frac{dM_y}{dt}$$
$$= -\frac{M_0 \mu_0 \omega}{2R}\cos(\omega t)$$

> **Number Time.** For $\mu_0=4\pi\times10^{-7}$ V·s/(A·m), $\omega=2\pi\cdot127$ MHz for protons at 3 Tesla, R = 15 cm (head coil), and $M_0 = 10^{-8}$ J/T (previously calculated magnetization of 1 mL of water), we get 30 μV, around the right order of magnitude for the voltages detected in magnetic resonance.

This is a small but detectable voltage level with today's electronics, and this is the basis of signal reception in modern MRI. The smaller the radius of the coils, R, the better: always build coils that are as small as possible! Furthermore, the signal is proportional to $\omega=\gamma B_0$, and increases with B_0 (although an exact analysis of the SNR will await a later chapter).

The Spins Can Be Excited With A Resonant RF Field

A Small RF Field Can Have a Large Effect if it is Resonant

The discussion of the previous section has shown that, in order to induce non-zero voltage, we must tilt the magnetization vector away from equilibrium and have it precess. Indeed, the basic MR experiment can be described as follows:

- **Thermal Equilibrium**: At thermal equilibrium, the spins are aligned along B_0 and do not precess.
- **Excitation**: The spins are somehow **excited**, that is, tilted to some angle θ with respect to B_0. This usually happens quickly and relaxation can be neglected.
- **Precession & Detection**: Once tilted, they precess and give off a time dependent magnetic field. The magnetic field induces a voltage in a nearby RF coil via Faraday's law. We can also further manipulate the spins with magnetic fields during this period to bring out particular contrast types. We usually have a time ~ T_2 before decoherence "eats up" the observable precessing magnetization.
- **Thermalization**: Relaxation processes kick in. The transverse magnetization decays with a time constant T_2 while the longitudinal magnetization builds up back up due to T_1 relaxation. If we wait for a time ≈ $5 \cdot T_1$, the magnetization will be back at its thermal equilibrium value.

Each such block (excite-acquire-wait) is called a **scan**. It is in fact not mandatory to wait for a time $5 \cdot T_1$ for the spins to return to thermal equilibrium; we'll see later on that waiting a shorter amount of time has both benefits (shorter scan times) and disadvantages (less signal per scan). For now, however, we'll assume that is the case, so M is equal to M_0 and points along the z-axis before the beginning of each scan.

We've already remarked that $B_{RF} \ll B_0$. How can we hope to non-negligibly excite the spins with such a weak RF field? The answer is that we use a **resonant field** that oscillates at the Larmor frequency. Namely, we are going to solve the Bloch equations setting **G**=0, and

$$\mathbf{B}_{RF} = B_1 \cos(\omega_{RF} t)\hat{\mathbf{x}} - B_1 \sin(\omega_{RF} t)\hat{\mathbf{y}} .$$

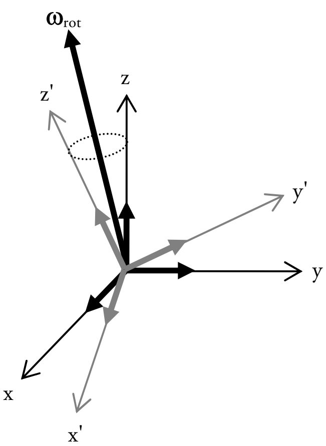
with

$$\omega_{RF} = \omega_0 \quad (\text{"on resonance irradiation"})$$

This means we will need to solve the Bloch equations with a time dependent magnetic field. Although a numerical solution is possible, we will employ a frame transformation trick which will enable us to solve this problem without any

Transforming to a Frame Which Rotates At The Same Frequency As The RF Field Makes it Appear Static: The Rotating Frame

In the laboratory frame, this amounts to solving the Bloch equations with a complicated time-dependent magnetic field. The Bloch equations are easier to solve in a frame which rotates around the z-axis with a frequency given by $\omega_{rot} = \omega_{RF}$. We tackle this as follows: consider a static (laboratory) frame with time independent, fixed unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, and a rotating frame with unit vectors $\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$.



If the rotating frame is rotating with an angular velocity ω_{rot} about an axis given by the unit vector $\hat{\mathbf{n}}$, then each of the axes of the rotating frame precess about the vector $\boldsymbol{\omega}_{rot} = \omega_{rot}\hat{\mathbf{n}}$. This means

Schematic for a simple single experiment ("scan")
(not drawn to scale)

Excite the spins. This usually takes << T_1, T_2 → Let spins precess while acquiring a signal until it decays due to T_2 → Wait for magnetization to "build back up" due to T_1 before exciting again → Time each obeys a precession equation identical (formally) to the Bloch equation:

$$\frac{d\hat{\mathbf{x}}'}{dt} = \hat{\mathbf{x}}' \times \boldsymbol{\omega}_{rot}$$
$$\frac{d\hat{\mathbf{y}}'}{dt} = \hat{\mathbf{y}}' \times \boldsymbol{\omega}_{rot}$$
$$\frac{d\hat{\mathbf{z}}'}{dt} = \hat{\mathbf{z}}' \times \boldsymbol{\omega}_{rot}$$

The magnetization vector can be expressed in either frame:

$$\begin{aligned}\mathbf{M}(t) &= M_x \hat{\mathbf{x}} + M_y \hat{\mathbf{y}} + M_z \hat{\mathbf{z}} \\ &= M_{x,rot}\hat{\mathbf{x}}' + M_{y,rot}\hat{\mathbf{y}}' + M_{z,rot}\hat{\mathbf{z}}'\end{aligned}$$

For example, if **B**(t) is

$$\mathbf{B}_{RF} = B_1 \cos(\omega_{RF} t)\hat{\mathbf{x}} - B_1 \sin(\omega_{RF} t)\hat{\mathbf{y}}$$

as given above, while the rotating frame rotates at the angular frequency $\omega_{rot}=\omega_{RF}$ about the z-axis ($\boldsymbol{\omega}_{rot}=\omega_{RF}\hat{\mathbf{z}}$) then the **components** of **B** in the two frames are:

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} B_1\cos(\omega_{RF}t) \\ -B_1 \sin(\omega_{RF}t) \\ B_0 \end{pmatrix} \qquad \begin{pmatrix} B_{x,rot} \\ B_{y,rot} \\ B_{z,rot} \end{pmatrix} = \begin{pmatrix} B_1 \\ 0 \\ B_0 \end{pmatrix}$$

Note the components change, but the vector is frame-independent since it is a geometrical quantity.

Differentiating **M**(t) with respect to time, we obtain

$$\begin{aligned}\frac{d\mathbf{M}}{dt} &= \frac{dM_{x,rot}}{dt}\hat{\mathbf{x}}' + \frac{dM_{y,rot}}{dt}\hat{\mathbf{y}}' + \frac{dM_{z,rot}}{dt}\hat{\mathbf{z}}' \\ &\quad + M_{x,rot}\frac{d\hat{\mathbf{x}}'}{dt} + M_{y,rot}\frac{d\hat{\mathbf{y}}'}{dt} + M_{z,rot}\frac{d\hat{\mathbf{z}}'}{dt} \\ &= \frac{dM_{x,rot}}{dt}\hat{\mathbf{x}}' + \frac{dM_{y,rot}}{dt}\hat{\mathbf{y}}' + \frac{dM_{z,rot}}{dt}\hat{\mathbf{z}}' \\ &\quad + \left(M_{x,rot}\hat{\mathbf{x}}' + M_{y,rot}\hat{\mathbf{y}}' + M_{z,rot}\hat{\mathbf{z}}'\right)\times\boldsymbol{\omega}_{rot} \\ &= \left(\frac{d\mathbf{M}}{dt}\right)_{rot} + \mathbf{M}\times\boldsymbol{\omega}_{rot}\end{aligned}$$

On the other hand, the Bloch equation says

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}.$$

Equating, we obtain:

$$\boxed{\left(\frac{d\mathbf{M}}{dt}\right)_{rot} = \gamma \mathbf{M} \times \left(\mathbf{B} - \frac{\boldsymbol{\omega}_{rot}}{\gamma}\right) \equiv \gamma \mathbf{M} \times \mathbf{B}_{eff}}.$$

This is precisely the Bloch equation but with an **effective field** $\mathbf{B}_{eff} = \mathbf{B} - \frac{1}{\gamma}\boldsymbol{\omega}_{rot}$.

The above equation is true for any rotating frame. However, in MRI, when we speak of "**the**" rotating frame, we will be referring to a frame which rotates at a constant angular velocity $\omega_{rot} = \omega_{RF}$ about the z-axis according to the left hand rule. For "the" rotating frame, $\boldsymbol{\omega}_{rot} = \omega_{rot}\hat{\mathbf{z}} = \omega_{RF}\hat{\mathbf{z}}$.

When expressed in the rotating frame, the components of the effective field $\mathbf{B}_{eff} = \mathbf{B} - \frac{1}{\gamma}\boldsymbol{\omega}_{rot}$ are:

$$\begin{pmatrix} B_{x,eff} \\ B_{y,eff} \\ B_{z,eff} \end{pmatrix} = \begin{pmatrix} B_1 \\ 0 \\ B_0 - \frac{1}{\gamma}\omega_{rot} \end{pmatrix}$$

(in the rotating frame: $\omega_{rot} = \omega_{RF}$)

If we select $\omega_{RF} = \gamma B_0 = \omega_0$ we are **on resonance**: the RF irradiates the spins at the same frequency as their natural frequency, ω_0. In this case:

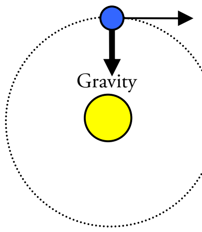
$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}$$

On resonance: $\omega_{RF} = \omega_0$

In the rotating frame: $\omega_{rot} = \omega_{RF}$

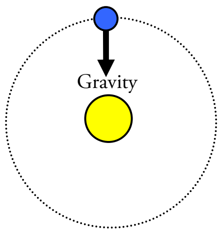
An Analogy From Mechanics

Imagine the earth going around the sun in a circle:

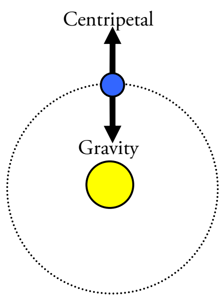


This can be understood by an observer in space the following way: the Earth wants to "go forward" but gravity pulls it "inward", curving its path into a circle. In effect, the Earth is continuously "falling" into the sun, but escaping doom thanks to its tangential velocity. All this is all a consequence of Newton's second law, F=ma.

Next, imagine how things would look to an observer standing on the sun and rotating with it. Neglecting for the time being the weather on the surface, the Earth would appear stationary to such an observer:



If that observer would try to use Newton's law F=ma to understand his world he would fail: according to $F=F_{gravity}=ma$, earth should be falling towards the sun, but it isn't! The truth is that when you transform to a rotating frame you need to add a **fictitious force**. That is, you need to pre-suppose a force which doesn't arise out of any physical source, called the centripetal force, to explain how it is possible for the earth to remain stationary:



So, in mechanics when you try to understand things in a rotating frame you need to do two things:

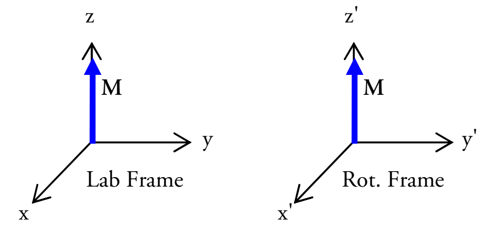
1. Understand how things in the "real" frame would look in the rotating frame (e.g., the Earth would remain still).
2. Add fictitious forces (e.g., the centripetal force).

A similar thing happens when you go to a rotating frame in magnetic resonance, rotating with the same angular velocity as the RF field:

1. First, the RF field appears stationary in the rotating field which "matches" its rotation frequency (i.e. because $\omega_{rot} = \omega_{RF}$).
2. Now we need to add the correct fictitious "force" - field, to be precise - given by $\mathbf{B}_{fict} = -\frac{1}{\gamma}\omega_{rot}$. To see, imaging a static spin in the lab frame, with no magnetic field. Now transform to a frame rotating with an angular velocity ω_{rot} about the z-axis. In this frame, the spin would appear to rotate with an angular velocity $-\omega_{rot} = \gamma B_{fict}$, as if there was a fictitious field $B_{fict} = -\frac{\omega_{rot}}{\gamma}$ present along the z-axis.

The Bulk Magnetization Precesses Around The Effective Field In The Rotating Frame

We've seen the magnetization vector obeys the Bloch equations in the rotating frame, only swapping the field for an effective field, $\mathbf{B}_{eff} = \mathbf{B} - \frac{1}{\gamma}\boldsymbol{\omega}_{rot}$ and expressing that field in the rotating frame basis (i.e. as it would appear to an observer rotating with the frame). This means **M** precesses about $\mathbf{B}_{eff}$ in the rotating frame. Starting from thermal equilibrium at time t=0, **M** points along B_0 (taken to coincide with the z-axis) in both the laboratory and the rotating frames, which are also assumed to coincide for t=0:

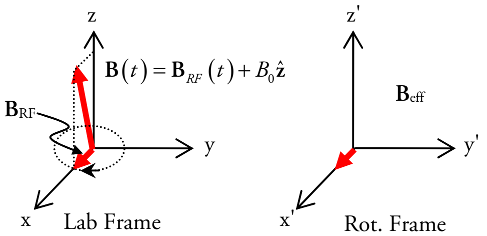


At time t=0 (thermal equilibrium), **M** points along the z-axis in both frames.

Now we turn on the resonant RF field in the laboratory frame:

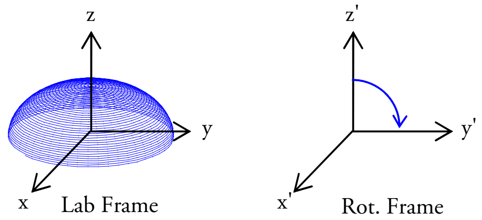
$$\mathbf{B}_{RF} = B_1 \cos(\omega_{RF} t)\,\hat{\mathbf{x}} - B_1 \sin(\omega_{RF} t)\,\hat{\mathbf{y}} \,.$$

This field rotates in the xy-plane in the lab frame, and appears stationary in the rotating frame. Furthermore, if we assume our irradiation is on resonance, $\omega_{RF}=\omega_0$, the effective field in the rotating frame has no z-component:



The magnetic field **B** in the laboratory frame has a large z-component and a small, rotating xy-component (not shown to scale). In the rotating frame, assuming B_{RF} is on resonance ($\omega_{RF}=\omega_0=\gamma B_0$) the effective field is static.

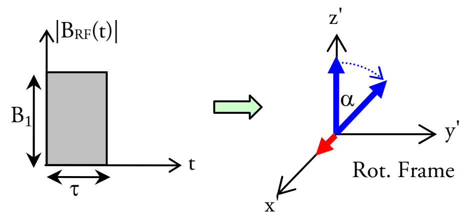
The magnetization **M** precesses about the x axis in the rotating frame. We can thus create any angle we'd like between it and the z-axis, depending on how long we let it precess and how strong B_1 is. Let's assume we have B_{RF} on for just enough time for the magnetization to tilt to the xy plane - that is, create a 90° angle between B_0 and **M**. Deducing the motion of **M** in the lab frame is now merely a matter of transforming back to the lab frame, which simply rotates at an angular velocity $-\omega_{rot}$ relative to the rotating frame. That is, **M** in the lab frame performs a spiral as it descends and rotates:



Shown here is the trajectory of the magnetization **M** in the lab (left) and rotating (right) frames. The two frames are connected by a simple rotation.

Setting The Radiofrequency (RF) Pulse's (Area)=(Duration)×(Amplitude) Sets The Flip Angle

We see the spins will perform a rotation about the x-axis in the rotating frame at a frequency $\omega_1=\gamma B_1$. Note this is **not the same as** ω_{RF}, (one is the amplitude of B_{RF}, the second is its oscillating frequency). After a time τ, **M** will have created an angle $\alpha = \omega_1 \tau = \gamma B_1 \tau$:



Note that

$$\alpha = \gamma\,(\text{amplitude of RF}) \times (\text{duration of RF})\,.$$

This relation is true only on resonance, when $\omega_{RF}=\omega_0$, where $\mathbf{B}_{eff}$ has no z-component.

To "tip" the magnetization onto the y axis, we wait a time t_{90} such that:

$$\alpha = \gamma B_1 t_{90} = \tfrac{\pi}{2}\,,$$

or

$$t_{90} = \tfrac{1}{4\gamma B_1}\,.$$

In the original laboratory (unrotating) frame the spins execute additional motions, but the important thing to realize is that **a spin which is in**

the xy plane in the rotating frame, must also be in the xy-plane in the laboratory frame (although *where* in the plane is a different story!).

Number Time. We've remarked that $B_{1,max}$ ~ 10 μT for an MRI scanner. For protons, one would need $t_{90} = \frac{\pi}{2\gamma B_1} \sim 0.5$ ms to excite the spins onto the xy-plane. For ^{13}C, $t_{90} = \frac{\pi}{2\gamma B_1} \sim 2$ ms .

Relaxation Can Be Neglected During Excitation Since Most Pulses Are Shorter Than T_1, T_2

Our calculations in the previous section have shown that excitation mostly happens on the timescale of milliseconds in MRI, which is much shorter than T_1, T_2. Hence, to an excellent approximation, relaxation effects can be neglected for most pulses and most tissue types in the body. We will make some remarks about the effects of relaxation later on but, in general, will neglect it unless specifically stated otherwise.

The Phase of the Pulse Determines The Phase of the Excited Magnetization

We have so far modeled B_{RF} in the lab frame as:

$$\mathbf{B}^{(RF,lab)} = \begin{pmatrix} B_1 \cos(-\omega_c t) \\ B_1 \sin(-\omega_c t) \\ 0 \end{pmatrix}.$$

Since we have full control over the x and y component we have no problem modulating both $B_1(t)$ and adding a time-dependent phase $\phi(t)$ to the RF field:

$$\mathbf{B}^{(RF,lab)} = \begin{pmatrix} B_1(t)\cos(-\omega_c t + \phi(t)) \\ B_1(t)\sin(-\omega_c t + \phi(t)) \\ 0 \end{pmatrix}$$

In the rotating frame, this will look like this[1]:

$$\mathbf{B}^{(RF,rot)} = \begin{pmatrix} B_1(t)\cos(\phi(t)) \\ B_1(t)\sin(\phi(t)) \\ 0 \end{pmatrix}.$$

Let's keep $B_1(t)$ and $\phi(t)$ fixed. Then the constant phase $\phi(t)=\phi_0$ is called the **phase of the pulse**, and is equal to the angle the RF field makes with the x-axis. determines where the RF pulse will point in the transverse plane.

The **phase of the magnetization** is defined as the angle made by the transverse component of the magnetization vector (i.e. its projection on the xy plane) with the x-axis.

Because the magnetization gets tipped at right angles to the RF field following the left hand rule, the relation between the pulse's and magnetization's phase is given by:

$$\phi_m = \phi_{RF} + \tfrac{\pi}{2}.$$

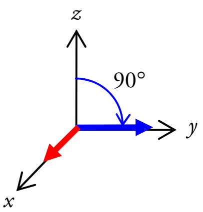
The standard notation for a constant RF pulse then assumes the form α_ϕ, where α is its flip angle and ϕ its (constant) phase. The following conventions are also used:

$$\begin{aligned} \phi &= 0^\circ: & x \\ \phi &= 90^\circ: & y \\ \phi &= 180^\circ: & -x \\ \phi &= 270^\circ: & -y \end{aligned}$$

Some examples are shown below (magnetization is assumed to start out from z, and is the blue vector; the RF is the red vector):

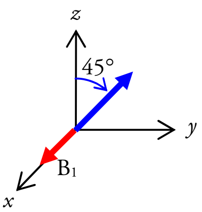
[1] To prove this, use $\mathbf{B}^{(rot)} = R_z(\omega_c t)\mathbf{B}^{(lab)}$, where $R_z(\omega_c t)$ is a RH rotation matrix about the z-axis by an angle $\theta = \omega_c t$ (the rotating frame rotates with a left handed rotation and angular frequency ω_c; in it, it appears the RF field rotates at the same angular frequency but in the opposite direction). There is a bit of algebra and trigonometry involved but the proof is straightforward.

$(90°)_x$ pulse



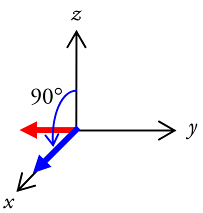
magnetization in yz plane (phase: 90°)

$(45°)_x$ pulse



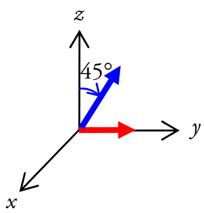
magnetization in yz plane (phase: 90°)

$(90°)_{-y}$ pulse



magnetization in xz plane (phase: 0°)

$(45°)_y$ pulse



magnetization in xz plane (phase: 180°)

Excitation Flip Angles < 90° Are Used To Minimize The Duration, Decrease Power Deposition At The Cost Of SNR

An excitation pulse need not tip the spins by 90°, and can create any angle α between **M** and the main $\mathbf{B}_0$ field. The disadvantage of this is its reduced signal: in our simple model we've seen that the voltage,

$$\varepsilon = -\frac{\mu_0}{2R}\frac{dM_y}{dt},$$

is proportional to the time derivative of M_y (reorienting the coil would introduce the time derivative of M_x, and would not change our conclusions). The magnitude of M_y will be proportional to the flip angle. Hence, the signal itself will also be proportional to sin(α) and decrease with the flip angle[2]:

$$\text{signal} \propto M_y, M_x \propto \sin(\alpha).$$

[2] We will see later on this decrease is actually mitigated in most sequences where pulses are applied rapidly (on the order of, or faster than T_1) and don't afford the magnetization enough time to return to thermal equilibrium before the next excitation.

On the other hand, the pulse's duration,

$$t_\alpha = \frac{\alpha}{\gamma B_1},$$

is proportional to the flip angle and decreases linearly (assuming we keep B_1 fixed).

Another advantage of short pulses is that they have reduced specific absorption rate (SAR). Some of the RF energy is absorbed in the patient's tissue and causes undesired heating. The amount of SAR is proportional to the square of B_1 and the pulse's duration:

$$\text{SAR} \propto \int_0^t \left|B_1(t)\right|^2 dt \overset{\substack{\text{constant}\\\text{pulse,}\\\text{duration T}}}{=} B_1^2 t_\alpha = \frac{\alpha B_1}{\gamma}.$$

We observe SAR reduces linearly with the flip angle. The amount of SAR is limited by most modern scanners' hardware based on our understanding of the effect of SAR on biological tissues. Modern RF coils deposit power on par with modern cell-phones and are generally considered safe as long as guidelines are observed.

Off-Resonant Excitation And The Concept Of Selective Excitation

The z-Field Can Vary as a Function of Position, Which Leads to Non-zero Offsets In The Rotating Frame

So far our approach has been to make B_0 disappear by moving to a rotating frame at a frequency $\omega_{rot} = \omega_0 = \gamma B_0$, in which the fictitious field negates B_0 completely:

$$B_z^{(lab)} = B_0 \rightarrow B_z^{(eff)} = B_0 - \tfrac{\omega_{rot}}{\gamma} = 0.$$

However, when B_0 varies as a function of position, $B_0 = B_0(\mathbf{r})$, it is impossible to make the z-component of the field disappear at every point:

$$B_z^{(lab)} = B_0(\mathbf{r}) \rightarrow B_z^{(eff)} = B_0(\mathbf{r}) - \tfrac{\omega_{rot}}{\gamma} \neq 0.$$

Some sources of variation could include:

1. Imperfections in the main magnet field.
2. Susceptibility artifacts in the sample, in which the external field induces microscopic magnetic moments which themselves distort the main field (in all directions, but predominantly in the direction on B_0).
3. Some patients might have metal implants which distort the magnetic field – again, in many directions, but their effect is most pronounced along B_0.
4. Often we <u>intentionally</u> create these inhomogeneities, as is the case with gradient coils, in which we create a linear dependence of the z-field on position:

$$B_0 \to B_0 + \mathbf{G}(t)\cdot\mathbf{r}\,.$$

For example, when a gradient is turned on,

$$B_z^{(lab)} = B_0 + \mathbf{G}(t)\cdot\mathbf{r} \to B_z^{(eff)} = \mathbf{G}(t)\cdot\mathbf{r}\,,$$

and, when the RF is turned on:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \mathbf{G}(t)\cdot\mathbf{r} \end{pmatrix}$$

If we have some form of spatial inhomogeneity due to hardware imperfections/susceptibility artifacts, we could write it as

$$B_z^{(lab)} = B_0 + \Delta B(\mathbf{r})\,,$$

and in the rotating frame its z-component will be

$$B_z^{(eff)} = B_z^{(lab)} - \tfrac{\omega_{rot}}{\gamma} = \Delta B(\mathbf{r})\,.$$

> <u>**Number Time**</u>. A gradient will create a range of frequencies given by $\gamma\!\!\!-G\Delta z$ over a spatial region of width Δz. Across a 1 mm pixel, this will be $\gamma\!\!\!-G\Delta z \approx 420$ Hz. Susceptibility artifacts at 3 Tesla will create spatial variations across the head on the order of hundreds of Hz, mostly in regions where air-tissue interfaces exist such as the prefrontal cortex, close to the oral cavity or ears, and so forth.

In the case the z-component is not completely zeroed out, we must analyze and understand the case for which

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix}.$$

The quantity ΔB will be referred to as the **offset** of the spins. Since we will be looking at a specific point in space we can assume ΔB is just a constant.

A Qualitative Solution: All Pulses Are Selective With A Finite Bandwidth Given By ~ $1/\gamma B_1$

It is fairly simple to divide our analysis into two extreme cases: in one, $\Delta B \ll B_1$, and we can neglect it, obtaining:

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} \approx \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}$$

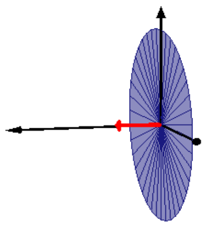
We thus recover the previous case in which we excite the spins "as usual", as if they were on resonance. On the other extreme, $\Delta B \gg B_1$,

$$\mathbf{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ \Delta B \end{pmatrix}.$$

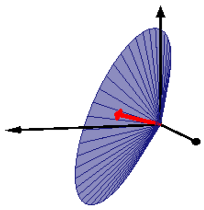
and the RF excitation will have no effect, resulting in no excitation. We can guess and extrapolate between these two extremes, saying that there is a cutoff to the effect of B_1 when $B_1 \sim \Delta B$. In other words, a range of offsets $\Delta B \sim B_1$ will be excited. This is known as the **bandwidth** of the pulse: the range of offsets (or frequencies) it will excite.

$$BW = \gamma\!\!\!-B_1\,.$$

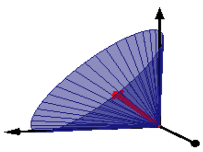
This can also be understood graphically, by plotting the precession cone of the spins about the effective field, starting out from thermal equilibrium (i.e. **M** along the z-axis):



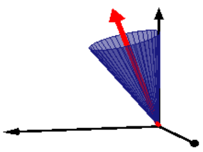
ΔB=0
B_{eff} points in the xy-plane – say, along the x-axis in this example - and the spin precesses in a circle in the yz plane.



ΔB<<B$_1$
B_{eff} starts tilting up in the xz plane, causing the rotation cone to start "folding".



ΔB=B$_1$
In this "dividing case", the x- and z- components of B_{eff} are equal. The "precession cone" just touches the xy plane.



ΔB>>B$_1$
B_{eff} now becomes close to the z-axis. The precession cone becomes very narrow: even if we wait for a long amount of time the spins will not stray far from the z-axis.

A Hard Pulse Is One With High Peak Power And An "Infinite" Bandwidth

When we are interested in flipping all of the spins onto the xy-plane regardless of their offset we must create a bandwidth larger than the range of offsets in our sample. Since $BW = \bar{\gamma} B_1$, this means we need to have a very high B$_1$>>range of offsets in our sample. The duration of an (on-resonance) 90°-pulse,

$$t_{90} = \frac{1}{4\bar{\gamma} B_1}.$$

We see that hard pulses are short and have a high peak power. Such pulses are called **hard pulses** in MR jargon. The "ideal" hard pulse is one for which B$_1$→∞, duration→0, such that $\gamma B_1 \cdot (\text{duration})$ equals the desired flip angle.

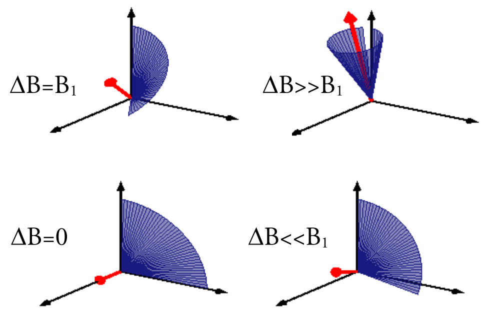
A Constant Pulse's Excitation Profile

Let us explore what happens when our pulse is not "hard" and has a finite duration. We'll look at a 90° pulse, although our conclusions will apply to any pulse flip angle.

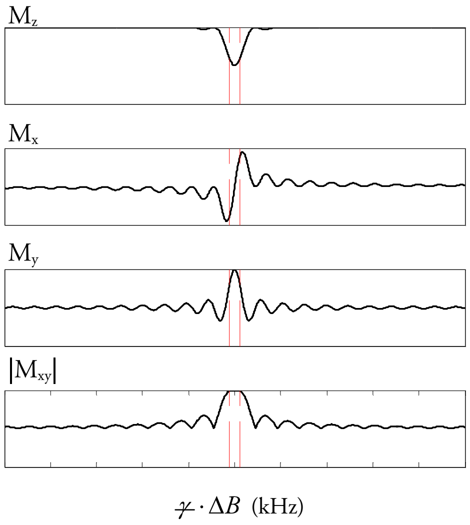
For a 90° flip angle, the duration must be

$$t_{90} = \frac{1}{4\bar{\gamma} B_1}.$$

However, as previously discussed, this only ensures a 90° flip angle for spins when ΔB=0. As we increase ΔB, the magnitude and direction of $\mathbf{B}_{eff}$ and its direction vary, and the corresponding final position of the magnetization - assumed to start out from thermal equilibrium along the z-axis - varies. In fact, even if B$_1$ is applied along x, spins not at the center do not even remain in the yz plane anymore. The following diagram shows the effective field and the precise trajectory traced by the magnetization vector during the pulse's duration, t_{90}, for the four cases outlined previously:



Instead of these pictorial diagrams, one can plot the components of **M** as a function of the offset, ΔB. This is known as the **pulse's response** or **pulse profile**. Such a response is plotted below for $\bar{\gamma} B_1 =$ 1 kHz (B$_1$≈23.5 μT), t_{90}=0.25 ms (B$_1$ applied along the x-axis as in the above diagrams, i.e. has 0° phase):



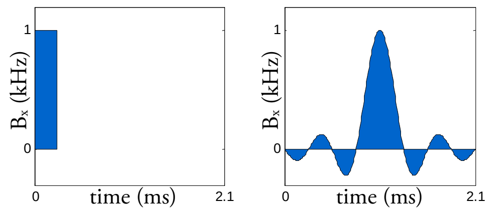
In the above, $\left|M_{xy}\right| = \sqrt{M_x^2 + M_y^2}$. The magnetization was assumed to have an arbitrary magnitude of unity, and the vertical axis stretches from ±1. The dashed red lines signify the points at which $\bar{\gamma}\Delta B = \bar{\gamma} B_1 = 1\ kHz$, which define the bandwidth of the pulse. It is quite clear that the concept of bandwidth has some artbitrariness to it since the profile of M_z and M_{xy} are not sharp.

The profile of M_{xy} actually looks somewhat wider than M_z, which is a result of the magnetization vector's constant magnitude, $|\mathbf{M}|^2 = |M_x|^2 + |M_y|^2 + |M_z|^2 = |M_{xy}|^2 + |M_z|^2 = 1$ and the relation between M_z and $\left|M_{xy}\right| = \sqrt{1 - |M_z|^2}$. For example, if $|M_z|$=0.9, then $|M_z|^2 = 0.81$ and $\left|M_{xy}\right| = 0.436$. So, even if $|M_z|$ is almost unperturbed, $|M_{xy}|$ might still appear to be quite sizable, leading to its wider profile.

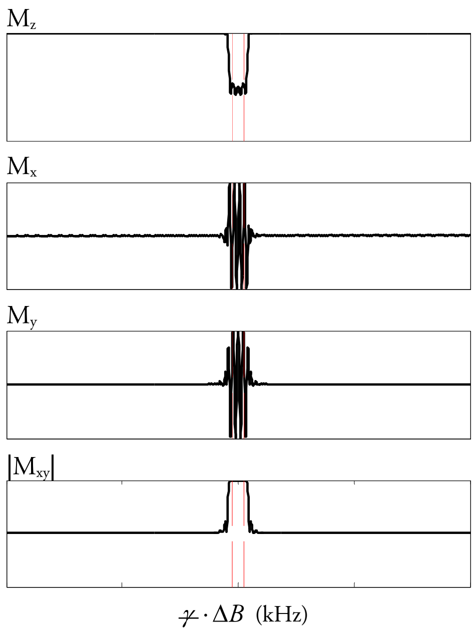
Also note the extensive "wiggles" in M_{xy} outside the slice, indicating that some excitation occurs even for $\Delta B \gg B_1$. We will deal with this in a moment by introducing shaped pulses.

Shaping The Pulse Affects The Pulse's Profile

Modern RF transmitters have the capability of shaping the RF pulse, $B_{RF}(t) = B_x(t) + iB_y(t)$; that is, controlling its x- and y- components. Such pulses are called **shaped pulses**. For the constant pulse we had $B_x = B_1$, $B_y = 0$. Let us see what happens if we vary $B_x(t)$ in a sinc-like manner:



The new pulse maintains the same peak B_1 and same area (and hence flip angle) as the rectangular pulse, but is necessarily longer (since the negative lobes of the sinc detract from the area). The frequency response of this pulse can be calculated by solving the Bloch equations numerically, yielding:



The response is shown using the same scaling and plot range as the rectangular pulse for a "fair" comparison. The dashed red lines represent the same bandwidth $(= \bar{\gamma} B_1)$ calculated for the rectangular pulse. The ensuing response is significantly better-behaved, with less wiggles and sharper transition lines. We won't go into the theory of shaped pulses in this course, but we will remark without proof that for tip angles up until about 90° the profile of M_{xy} resembles the Fourier transform of $B_{RF}(t)$.

The Slice's Center Can Be Shifted By Sinusoidally Modulating The Pulse's Phase

The pulses discussed so far excite a bandwidth about a central frequency $\not{\gamma}\Delta B = 0$. There is a very simple way to shift the center of the excited slice, by modulating the pulse's phase with a linear term. Mathematically, this means:

$$\mathbf{B}_{\text{eff}} = \begin{pmatrix} B_1 \\ 0 \\ \Delta B \end{pmatrix} \to \begin{pmatrix} B_1 \cos(-\omega_c t) \\ B_1 \sin(-\omega_c t) \\ \Delta B \end{pmatrix} \equiv \mathbf{B}_{\text{eff}}^{(\text{shift})}.$$

To understand why this shifts the pulse, imagine being given $\mathbf{B}_{\text{eff}}^{(\text{shift})}$ in the rotating frame, where it rotates around the z-axis according to the left hand rule with angular frequency ω_c. By performing a *second* rotating frame transformation, into a frame which rotates with ω_c relative to the original ("first") rotating frame, we fix B_1 and add an additional fictitious field:

$$\mathbf{B}_{\text{eff}}^{(\text{shift})} \xrightarrow[\text{frame}]{\text{2nd rotating}} \begin{pmatrix} B_1 \\ 0 \\ \Delta B - \frac{\omega_c}{\gamma} \end{pmatrix}.$$

In this 2nd rotating frame the z-component has a fixed offset. All of our previous arguments can be repeated, but now the center of the profile would not occur at $\not{\gamma}\Delta B = 0$, but rather at a frequency defined by $\Delta B - \frac{\omega_c}{\gamma} = 0$, or:

$$\not{\gamma}\Delta B = \frac{\omega_c}{2\pi}.$$

Current MRI hardware enables one to control the phase and amplitude of the RF pulse as a function of time, making shifting the profile's center easy (we can generate any practical frequency ω_c). It also places almost no demands on the hardware[3].

It's interesting to note that, using complex notation, our constant original field

$$B_{xy} = B_x + iB_y = B_1$$

becomes

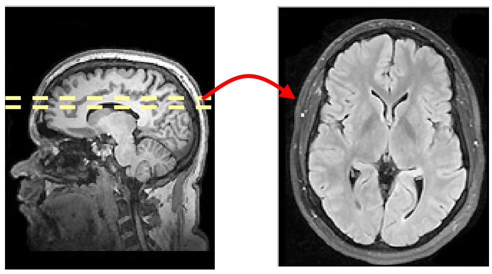
$$\begin{aligned} B_{xy} &= B_x + iB_y \\ &= B_1 \cos(-\omega_c t) + iB_1 \sin(-\omega_c t) \\ &= B_1 \left(\cos(-\omega_c t) + i \sin(-\omega_c t)\right) \\ &= B_1 e^{-i\omega_c t} \end{aligned}$$

The complex notation makes the effect seem much simpler (i.e. only multiplying the complex B_{RF} waveform by $e^{-i\omega_c t}$).

Slice Selection

Often, We Are Interested In Exciting A Single Slice

We now come to the first form of spatial selectivity in MRI: selective excitation, in which only a part of the sample is excited. We will confine ourselves to the simple scenario of one-dimensional excitation, meaning selectively exciting a range of positions along a fixed axis:



Without such selective excitation, an RF pulse would excite the entire sample. Although this can be and is sometimes done, a slice-selective approach also has its own merits.

Applying A Pulse In The Presence Of A Gradient Will Excite A Slice

In the joint presence of a gradient and an RF irradiation, the effective field in the rotating frame is:

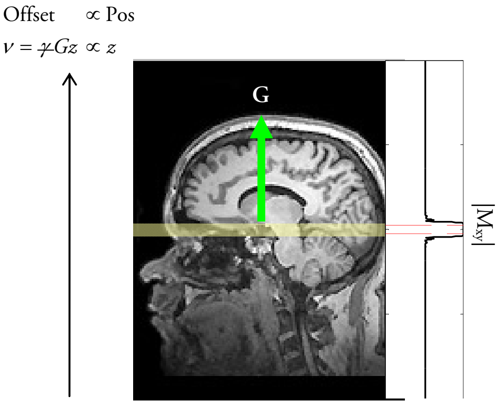
$$\mathbf{B}_{\text{eff}}(\mathbf{r},t) = \begin{pmatrix} B_1 \\ 0 \\ \mathbf{G}(t)\cdot\mathbf{r} \end{pmatrix}.$$

[3] This "trick" for shifting the pulse's profile and the relevant analysis outlined, also holds for non-constant pulses.

We will assume for simplicity our gradient is constant and turned on along the z-direction, so $\mathbf{G}(t) = G\hat{\mathbf{z}}$, and so:

$$\mathbf{B}_{\text{eff}}(z) = \begin{pmatrix} B_{RF,x}(t) \\ B_{RF,y}(t) \\ Gz \end{pmatrix}.$$

The gradient creates a linearly increasing offset along z:

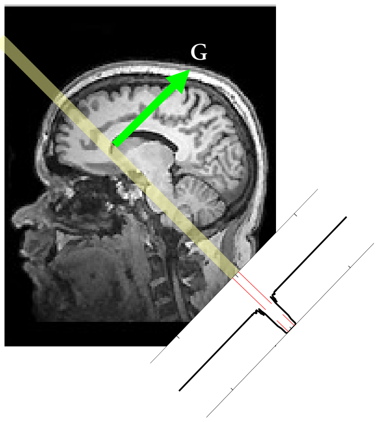


The gradient assigns frequencies to positions via $\nu = \not\gamma Gz$, and hence any pulse that excites a range of frequencies $BW = \not\gamma B_1$ will, in the presence of a gradient, excite a range of positions given by:

$$\Delta z = \frac{BW}{\not\gamma G} = \frac{B_1}{G}$$

This is, in fact, the **slice thickness**.

The excited slice will be perpendicular to the direction of the gradient vector **G**. For example applying the same pulse with a gradient in a different direction - say, at 45° to the z-axis - will excite a slice that is itself tilted by 45°, since now our gradient will create a linear correspondence between frequency and the x+z=const planes:



Finally, just as the profile of a pulse can be shifted in frequency space as a function of the offset, thus the slice's center can also be shifted by simply modulating the RF pulse's shape:

Acquisition

The Time Evolution Of The Magnetization In The Rotating Frame In The Absence Of RF Irradiation

In the absence of RF irradiation, the Bloch equations consist only of a z-field, made up of its position-dependent offset due to gradients, inhomogeneities and so forth:

$$\gamma \mathbf{B}_{\text{eff}} = \begin{pmatrix} 0 \\ 0 \\ \gamma \Delta B(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega(\mathbf{r}) \end{pmatrix}.$$

We would like to solve the Bloch equations:

$$\begin{cases} \dot{M}_x = \gamma\left(M_y B_{\text{macro},z} - M_z B_{\text{macro},y}\right) - \frac{M_x}{T_2} \\ \dot{M}_y = \gamma\left(M_z B_{\text{macro},x} - M_x B_{\text{macro},z}\right) - \frac{M_y}{T_2} \\ \dot{M}_z = \gamma\left(M_x B_{\text{macro},y} - M_y B_{\text{macro},x}\right) - \frac{M_z - M_0}{T_1} \end{cases}$$

Substituting the field above, we obtain:

$$\begin{cases} \dot{M}_x = M_y \omega(\mathbf{r}) - \frac{M_x}{T_2} \\ \dot{M}_y = -M_x \omega(\mathbf{r}) - \frac{M_y}{T_2} \\ \dot{M}_z = -\frac{M_z - M_0}{T_1} \end{cases}$$

The first thing to note is that the transverse (xy) and longitudinal (z) components of **M**(t) are **decoupled**: the transverse magnetization precesses in the xy-plane and decays with a time constant T_2, while the longitudinal component builds up towards thermal equilibrium with a time constant T_1, and is unaffected by the z-component of the field. We can solve for M_z independent of the gradients, field inhomogeneities, etc ... the solution having been already introduced in lecture 4:

$$\boxed{M_z(t) = M_z(0)e^{-t/T_1} + \left(1-e^{-t/T_1}\right)M_0}$$

The equations for M_x and M_y are mixed together, or **coupled** in mathematical terms. We will therefore treat the transverse magnetization (M_x and M_y) as one entity:

$$M_{xy} = M_x + iM_y .$$

We now multiply the second equation by *i* and add up the first two equations:

$$\frac{dM_x}{dt} + i\frac{dM_y}{dt} = \omega(\mathbf{r},t)\left(M_y - iM_x\right) - \frac{M_x + iM_y}{T_2}$$

The LHS is merely the time derivative of M_{xy}. Similar simplifications can be made for the RHS, and we obtain:

$$\frac{dM_{xy}}{dt} = \left\{-i\omega(\mathbf{r},t) - \frac{1}{T_2}\right\}M_{xy} .$$

This equation is of the form

$$\frac{dy}{dt} = a(t)y$$

with

$$a(t) = -i\omega(\mathbf{r},t) - \frac{1}{T_2} .$$

If *a* were constant, its solution would be simple: $y(t) = y(0)e^{at}$. Once *a* is time dependent we must break down the time interval into small steps Δt such that *a*(t) is constant in each time step. Then the solution in each interval is

$$y(t+\Delta t) = e^{a(t)\Delta t}y(t) .$$

The full solution is obtained by concatenating the short-time solutions:

$$\begin{aligned}
y(\Delta t) &= y(0)e^{a(0)\Delta t} \\
y(2\Delta t) &= y(\Delta t)e^{a(\Delta t)\Delta t} &&= y(0)e^{(a(\Delta t)+a(0))\Delta t} \\
y(3\Delta t) &= y(2\Delta t)e^{a(2\Delta t)\Delta t} &&= y(0)e^{(a(2\Delta t)+a(\Delta t)+a(0))\Delta t} \\
y(4\Delta t) &= y(3\Delta t)e^{a(3\Delta t)\Delta t} &&= y(0)e^{(a(3\Delta t)+a(2\Delta t)+a(\Delta t)+a(0))\Delta t} \\
&\ldots
\end{aligned}$$

This can be continued by induction, with the sum turning into an integral as $\Delta t \to 0$:

$$y(t) = e^{\int_0^t a(t')dt'}y(0) .$$

Returning back to our case and substituting the expression for a(t)= $\omega(\mathbf{r},t)$, we obtain:

$$\boxed{M_{xy}(\mathbf{r},t) = M_{xy}(\mathbf{r},0)e^{-\int_0^{t'}\omega(\mathbf{r},t)dt' - t/T_2}}.$$

Usually, t=0 will correspond to the point in time right after the excitation pulse, so $M_{xy}(\mathbf{r},0)$ will be the transverse magnetization right after excitation.

Note that we haven't really said if we're in the rotating or lab frame, and our derivation - and final expression - are valid for both. Since $\omega^{(rot)}(\mathbf{r},t) = \omega^{(lab)}(\mathbf{r},t) - \omega_0$, we can also deduce that

$$M_{xy}^{(lab)}(\mathbf{r},t) = e^{-i\omega_0 t}M_{xy}^{(rot)}(\mathbf{r},t)$$

which makes sense, because the lab and rotating frames differ by a constant rotation about the z-axis by an angular frequency ω_0.

The Acquired Signal In MRI Is Proportional To The Transverse Magnetization

We now state a very fundamental relation in MRI: the acquired signal is proportional to the integral of the magnetization, weighted by the receiver profile:

$$\boxed{s(t) \propto \omega_0 \int_{\text{body}} \overline{B_{xy}^{(rec)}(\mathbf{r})} M_{xy}^{(\text{rot})}(\mathbf{r},t)\, dV}$$

where $B_{xy}^{(rec)}$ is the spatial field dependence of the receiver coil when we drive a unit current through it. In the remainder of this section we derive this equation, but you can skip this without loss of continuity.

__Proof__: Armed with our expression for the time course of the magnetization, we turn to deriving a usable expression for the acquired signal (voltage) in our receiver coil. We've seen in Lecture 3 that, for any receiver coil,

$$\begin{aligned} v_{rec} &= -\int_{\text{body}} \mathbf{B}_{rec}(\mathbf{r}) \cdot \frac{d\mathbf{M}(\mathbf{r},t)}{dt} dV \\ &= -\int_{\text{body}} \left[B_x^{(rec)} \frac{dM_x}{dt} + B_y^{(rec)} \frac{dM_y}{dt} + B_z^{(rec)} \frac{dM_z}{dt} \right] dV \end{aligned}$$

.

Here **M** and $\mathbf{B}_{\text{rec}}$ are both measured in the laboratory frame. First, the time derivative of the z-component of the magnetization, which changes with a time constant T_1 (on the order of Hz) is much smaller than the x- and y- components, which precess with a frequency of MHz. Therefore, we can neglect the z-component to an excellent approximation:

$$v_{rec} = -\int_{\text{body}} \left(B_x^{(rec)} \frac{dM_x}{dt} + B_y^{(rec)} \frac{dM_y}{dt} \right) dV .$$

We define $M_{xy} = M_x + iM_y$, $B_{xy} = B_x + iB_y$ and note:

$$\begin{aligned} \overline{B_{xy}} M_{xy} &= (B_x - iB_y)(M_x + iM_y) \\ &= B_x M_x + B_y M_y + i(B_x M_y - M_x B_y) \end{aligned}$$

so

$$B_x M_x + B_y M_y = \text{Re}\left(\overline{B_{xy}} M_{xy}\right).$$

Therefore:

$$v_{rec} = -\text{Re}\int_{\text{body}} \overline{B_{xy}^{(\text{rec})}} \frac{dM_{xy}^{(\text{lab})}}{dt} dV .$$

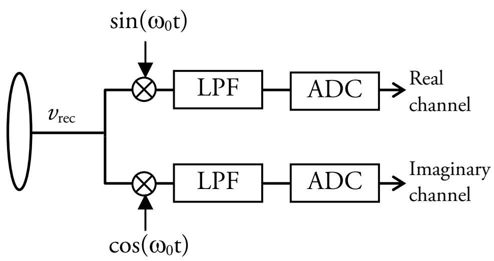
Since we have an expression for M_{xy} in the lab frame, we can differentiate it. Since ω_0 varies much faster than T_2 and $\gamma\mathbf{G}(t)\cdot\mathbf{r}$, we can neglect both terms and obtain[4]:

$$\frac{dM_{xy}^{(\text{lab})}(\mathbf{r},t)}{dt} \approx -i\omega_0 M_{xy}^{(\text{lab})}(\mathbf{r},t)$$

so

$$v_{rec} = \omega_0 \,\text{Re} \int_{\text{body}} i\overline{B_{xy}^{(\text{rec})}(\mathbf{r})} M_{xy}(\mathbf{r},t)\, dV$$

The interesting signal is modulated by a rapid (~100 MHz) phase term, $e^{-i\omega_0 t}$, which is "hidden" inside M_{xy}. It is beneficial to get rid of it for both convenience, as well as to lessen the burden on the analog-to-digital converter (which needs to deal with slower varying signals). This is called **demodulation**. To do this, v_{rec} is split into the identical copies, with one multiplied by $\cos(\omega_0 t)$ and the other by $\sin(\omega_0 t)$, and each is then passed through a low-pass filter (LPF)



Using

$$\begin{aligned} \cos(-\omega_0 t) &= \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \\ \sin(-\omega_0 t) &= \frac{e^{-i\omega_0 t} - e^{i\omega_0 t}}{2i} \end{aligned}$$

and writing

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[4] This assumption needs to be modified when studying solids with very short T_2s (on the order of microseconds). However, such short T_2s are rarely observed in MRI since they lead to signals well below the noise levels.

$$M_{xy}^{(\mathrm{lab})}(\mathbf{r},t) = \left|M_{xy}^{(\mathrm{lab})}(\mathbf{r},t)\right| e^{i\phi_M(\mathbf{r},t)}$$
$$B_{xy}^{(rec)}(\mathbf{r},t) = \left|B_{xy}^{(rec)}(\mathbf{r},t)\right| e^{i\phi_B(\mathbf{r},t)}$$

we obtain, right before the LPF:

$$(v_{rec})_{im} = \tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} i\left|B_{xy}\right|\left|M_{xy}\right|\left[e^{i[\omega_0 t+\phi_M-\phi_B]} + e^{i[-\omega_0 t+\phi_M-\phi_B]}\right]dV$$
$$(v_{rec})_{re} = \tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} \left|B_{xy}\right|\left|M_{xy}\right|\left[e^{i[-\omega_0 t+\phi_M-\phi_B]} - e^{i[\omega_0 t+\phi_M-\phi_B]}\right]dV$$

Since the LPF removes the fast changing component $-\omega_0 t+\phi_M-\phi_B$, we obtain, after the LPFs:

$$\begin{aligned}(v_{rec})_{im} &= \tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}} i\left|B_{xy}\right|\left|M_{xy}\right| e^{i[\omega_0 t+\phi_M-\phi_B]}dV \\ &= -\tfrac{\omega_0}{2}\int_{\mathrm{body}}\left|B_{xy}\right|\left|M_{xy}\right|\sin(\omega_0 t+\phi_M-\phi_B)\,dV \\ (v_{rec})_{re} &= -\tfrac{\omega_0}{2}\,\mathrm{Re}\int_{\mathrm{body}}\left|B_{xy}\right|\left|M_{xy}\right| e^{i[\omega_0 t+\phi_M-\phi_B]}dV \\ &= -\tfrac{\omega_0}{2}\int_{\mathrm{body}}\left|B_{xy}\right|\left|M_{xy}\right|\cos(\omega_0 t+\phi_M-\phi_B)\,dV\end{aligned}$$

We then form the complex signal in the computer:

$$\begin{aligned}s(t) &= (v_{rec})_{re} + i(v_{rec})_{im} \\ &= -\tfrac{\omega_0}{2}\int_{\mathrm{body}}\left|B_{xy}^{(rec)}\right|\left|M_{xy}^{(\mathrm{lab})}\right| e^{i(\omega_0 t+\phi_M-\phi_B)}dV\end{aligned}$$

Since $\left|M_{xy}^{(\mathrm{lab})}\right| = \left|M_{xy}^{(\mathrm{rot})}\right|$, and since the rotating and lab frame magnetization vectors are related via $M_{xy}^{(\mathrm{rot})}(\mathbf{r},t) = M_{xy}^{(\mathrm{lab})}(\mathbf{r},t)e^{i\omega_0 t}$, we can simplify:

$$s(t) \propto \omega_0 \int_{\mathrm{body}} \overline{B_{xy}^{(rec)}(\mathbf{r})}\, M_{xy}^{(\mathrm{rot})}(\mathbf{r},t)\,dV$$

We've omitted the constant of proportionality since the actual measured signal's magnitude will depend anyway on the electronics, amplifiers and so on.