Lecture 5
Basic Imaging

Lecture Notes by Assaf Tal

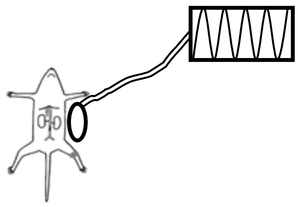
Frequency Encoding

We Cannot Resolve Our Signal Spatially Without A Gradient

Since MRI happens in the near field we have no spatial control over our fields. As we've previously seen, the acquired signal is given by:

$$s(t) \propto \omega_0 \int_{\text{body}} \overline{B_{xy}^{(rec)}(\mathbf{r})} M_{xy}^{(\text{rot})}(\mathbf{r},t)\, dV \, .$$

We could get some selectivity by shaping the field of our receiver. For example, in the early days of MR people would acquire a signal by placing a coil close to the object of interest:



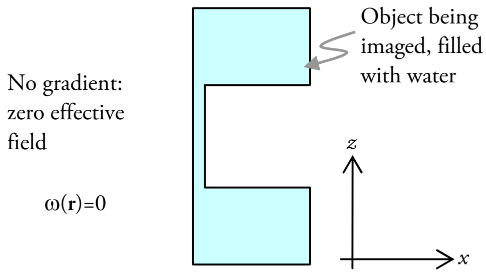
By placing a coil close to the rat's kidney, one can pick up signal mostly from the kidney where $\mathbf{B}_{rec}$ is strongest.

This is highly inefficient, the reception pattern is spatially inhomogeneous (since $\mathbf{B}_{rec}$ is inhomogeneous) and it requires one to mechanically move the coil (or subject) to change their sampling point.

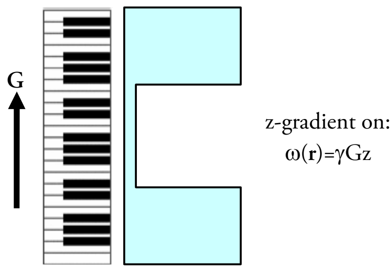
In MRI one takes a different approach: first, the receiver's field is made as homogeneous as possible over the object, and gradient fields are used to spatially resolve our signal through one of two methods: frequency encoding or phase encoding. We describe them in order.

Pictorial Description Of Frequency Encoding

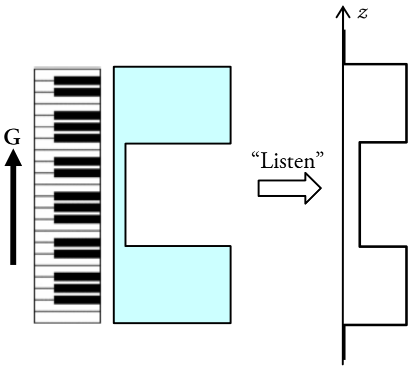
Imagine exciting the spins in the object onto the xy-plane. In the absence of a gradient, all the spins see the same field and precess at the same frequency. Imagine being able to "listen" to their frequencies: you would hear one well defined tone.



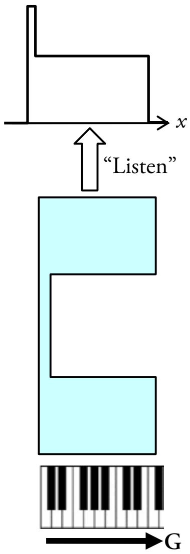
Once you turn on a gradient, say, along the z-axis, we generate a different frequency along each point on the z-axis. You can imagine each position being assigned a different "key" on a "piano":



The intensity we "hear" at frequency - at each piano key - is proportional to the number of spins at that position. Therefore, **by "listening" to the signal we can deduce the distribution of spins along the z-axis**. This is the basic idea behind frequency encoding. The image obtained would give us a **projection** of the density of spins on the z-axis:

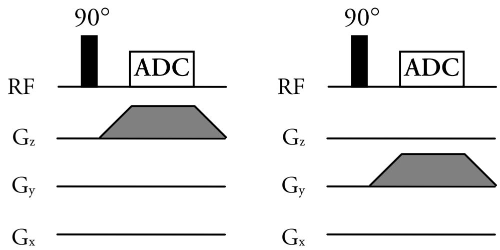


By changing the direction of the gradient we alter the axis of projection, which is parallel to **G**:



The Frequency Encoding Pulse Sequence

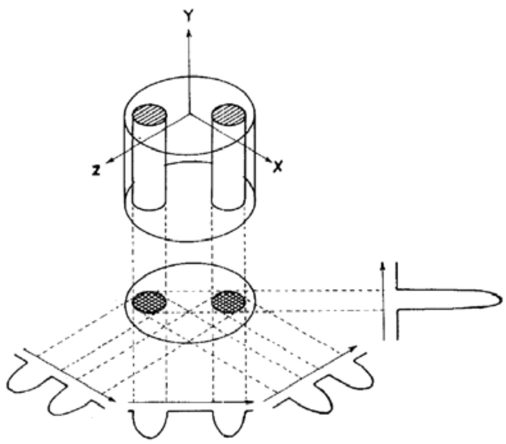
The frequency encoding experiment can be described with a **pulse sequence**, which is a diagram indicating the timing and amplitudes of the RF and gradient channels, as well as delays and acquisition blocks:



Left: pulse sequence depicting a volume excitation of the entire sample (i.e. not a slice selective excitation), followed by frequency encoding along z. Right: same sequence, only with frequency encoding along y.

Multiple Frequency Encoding Scans Can Be Used To Reconstruct An Image Via Projection-Reconstruction

Although it might be unclear how to do this, the reader might feel that, given 1D projections along enough axes, one could infer the 3D spatial distribution of spins in the sample. This is correct, and is known as **projection reconstruction**. This is how Computerized Tomography (CT) scanners work. Some MRI experiments do use this approach, the most famous being the nobel prize winning paper[1] by Paul Lauterbur which first introduced this concept and kickstarted the MRI field, where the first figure shows projections of two test tubes filled with water:



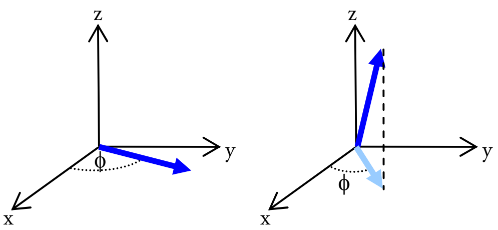
The reader is encouraged to dig up the paper and enjoy this piece of scientific history in the making.

[1] PC Lauterbur, Nature 242:190-191 (1973)

Phase Encoding

A Precessing Spin Has A "Phase"

The phase of a precessing spin is defined as the phase its projection on the xy-plane makes with the x-axis:



Left: the phase of a spin in the xy-plane is the angle it makes with the x-axis. Right: for a spin not in the xy-plane, one examines the phase its projection on the xy-plane makes with the x-axis.

One can think of this mathematically as follows: suppose you are given a magnetization vector

$$\mathbf{M} = \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix}.$$

Its projection on the xy plane is the two dimensional vector

$$\begin{pmatrix} M_x \\ M_y \end{pmatrix}.$$

One can form the complex quantity $M_{xy} = M_x + iM_y = |M_{xy}|e^{i\phi}$. The phase ϕ of this quantity is precisely the phase of the spin.

A spin precessing with a constant angular frequency ω for a time *t* will accumulate a phase:

$$\phi(t) = \omega t .$$

If ω is time dependent, one must break down the time into small intervals Δt during which ω is approximately constant, and sum up the phase contributions from each:

$$\phi = \omega(t_1)\Delta t + \omega(t_2)\Delta t + \ldots \omega(t_N)\Delta t .$$

As $\Delta t \to 0$ this becomes an integral:

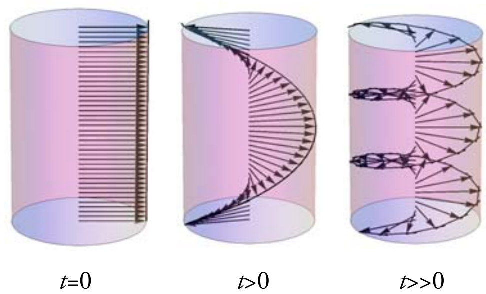
$$\phi(t) = \int_0^t \omega(t')dt' .$$

Precession In The Presence Of A Gradient Creates A "Spin-Helix" Along The Gradient Axis

Once a gradient is turned on, the frequency becomes spatially dependent, and so does the phase of the spins. For a z-gradient, $\omega = \gamma G z$ and:

$$\phi(t) = \gamma G z t .$$

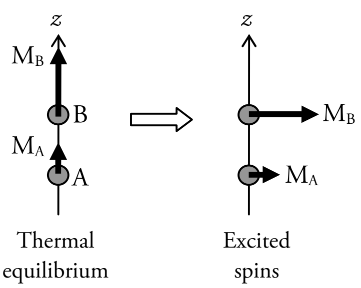
One should visualize this as a "helical winding" of the spins along the gradient axis:



In the presence of a z-gradient, the spins precess with a frequency ω=γGz which induces a linear phase ϕ(t)=γGzt, imparting a helical shapre to the tips of the magnetization vectors as time progresses.

The Principle Of Phase Encoding (PE)

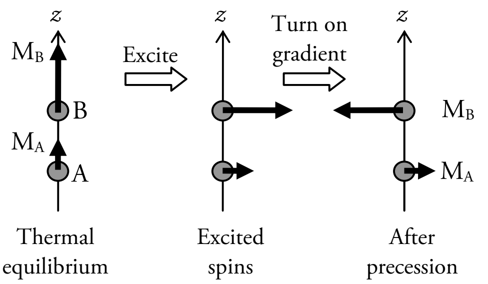
Imagine having a different density of spins at two points, A and B, along the z-axis. Our task is to deduce the density of spins at the two points, M_A and M_B:



By exciting and acquiring, the signal we would measure would originate from both points equally, and would lack any spatial selectivity:

$$s_1 \propto M_A + M_B\,.$$

We now run a second experiment, in which we apply a gradient just for long enough for the spins to go out of phase (let's assume for simplicity that A corresponds to z=0, so the spin there is stationary):



At this point when we acquire a signal, it will be proportional to

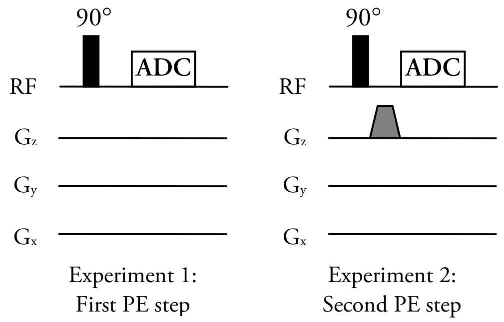
$$s_2 \propto M_A - M_B$$

By adding and subtracting the two experiments, one can extract just the signal from A, or just the signal from B:

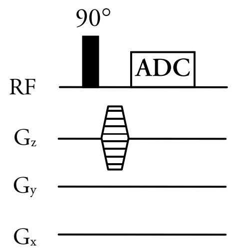
$$s_1 + s_2 = M_A$$
$$s_1 - s_2 = M_B$$

This idea can be extended to more than just two positions and more than one dimension: **by performing multiple experiments, creating a unique phase distribution in each experiment (using the gradients) and taking linear combinations of those experiments, the signal from multiple points in the sample can be recovered**. That is the principle of phase encoding.

In terms of pulse sequences, the two experiments would look like this:



These are often notated as:



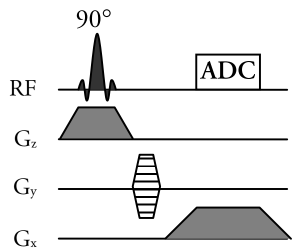
The striped gradient shape indicates that is a phase encoding gradient, which should be incremented by some fixed amount between successive phase encoding steps.

In the above example we needed 2 expeirments to phase encode two spatial points, A and B. In general, one would need N experiments to retrieve N spatial points. This is true even if the points are in 3D.

Number time. The human head can be enclosed in a box about 20×18×16 cm. If we wanted a spatial resolution of 1 mm^3, we would need $200\times180\times160\approx6\cdot10^6$ voxels, which would require about half a million scans using phase encoding! As the sole method of imaging, phase encoding is unsuitable for high resolution spatial imaging.

Phase Encoding Is Often Combined With Frequency Encoding And Slice Selection Along Orthogonal Axes

To save time, frequency and phase encoding are often combined: a single frequency encoding experiment – which produces a 1D projection – is repeated with phase encoding along an orthogonal axis, all inside a thinly excited slice. In terms of pulse sequences, we might have:



K-Space: A Mathematical Description Of Signal Encoding In MRI

The Time Evolution Of The Magnetization In The Presence of Gradients

We've previously seen that our acquired signal is

$$s(t) \propto \omega_0 \int_{\text{body}} \overline{B_{xy}^{(rec)}(\mathbf{r})} M_{xy}^{(\text{rot})}(\mathbf{r},t)\, dV$$

with (omitting the *rot* superscript for simplicity):

$$M_{xy}(\mathbf{r},t) = M_{xy}(\mathbf{r},0)\, e^{-\int_0^{t'} \omega(\mathbf{r},t) dt' - t/T_2}.$$

Neglecting field inhomogeneities for the time being, and in the absence of RF irradiation, the effective field in the rotating frame in the presence of gradients is:

$$\mathbf{B}_{\text{eff}} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{G}(t)\cdot\mathbf{r} \end{pmatrix}.$$

Thus, $\omega(\mathbf{r},t) = \gamma \mathbf{G}(t)\cdot\mathbf{r}$. Substituting into M_{xy}, we obtain:

$$M_{xy}(\mathbf{r},t) = M_{xy}(\mathbf{r},0)\, e^{-\gamma\left(\int_0^{t'} \mathbf{G}(t')dt'\right)\cdot\mathbf{r} - t/T_2}.$$

We now **define** a new variable, which will be of great importance in future discussions (recall $\gamma = 2\pi\bar{\gamma}$):

$$\boxed{\mathbf{k}(t) = \bar{\gamma}\int_0^t \mathbf{G}(t')\, dt'}$$

such that:

$$\boxed{M_{xy}(\mathbf{r},t) = M_{xy}(\mathbf{r},0)\, e^{-2\pi i \mathbf{k}(t)\cdot\mathbf{r}} e^{-t/T_2}}$$

In the last step we rewrote $M_{xy}(\mathbf{r},0)$ using its magnitude and phase:

$$M_{xy}(\mathbf{r},0) = \left|M_{xy}(\mathbf{r},0)\right| e^{i\phi_0(\mathbf{r})}.$$

The MRI Signal Is Acquired In The Fourier Space ("k-Space") Of The Image

Making use of the signal expression from the previous section and substituting our expression for $M_{xy}^{(\text{rot})}$, we obtain:

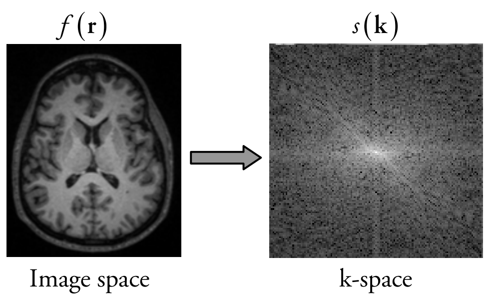
$$s(t) \propto \omega_0 \int_{\text{body}} \overline{B_{xy}^{(rec)}(\mathbf{r})} M_{xy}^{(\text{rot})}(\mathbf{r},0)\, e^{-2\pi i \mathbf{k}(t)\cdot\mathbf{r}} e^{-t/T_2(\mathbf{r})} dV$$

For the time being we will assume our acquisition takes much less than T_2, so we can neglect its effect. Our signal then assumes the following general form:

$$s(t) = s(\mathbf{k}(t)) = \int_{\text{body}} f(\mathbf{r})\, e^{-2\pi i \mathbf{k}(t)\cdot\mathbf{r}} dV.$$

Once we neglect T_2, the signal depends on time only via **k**(t), and so is really a function of **k**. Thus, we can think of s(**k**(t)) as being acquired in some 2D or 3D "k-space".

Given f(**r**), it is computationally straightforward to compute s(**k**), and a sample calculation is shown below:



The question we would like to pose is the opposite: given s(**k**), can we recover f(**r**)? The answer is yes – to an extent – and is related to perhaps the most famous transform in mathematics, the Fourier transform.

k-Space And Image Space Are Related Via A Continuous Fourier Transform

Given a function g(**r**), we can **define** its **continuous Fourier transform** (CFT) h(**k**) as:

$$h(\mathbf{k}) = \int_{-\infty}^{\infty} g(\mathbf{r}) e^{-2\pi i \mathbf{k}\cdot\mathbf{r}} d\mathbf{r}\,.$$

This is just a **definition**. However, it is now possible to prove that, if g(**r**) is sufficiently smooth and well behaved, that it is given by the inverse continuous fourier transform (ICFT) of h(**k**):

$$g(\mathbf{r}) = \left(\tfrac{1}{2\pi}\right)^3 \int h(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{r}} d\mathbf{k}\,.$$

There are multiple proofs of this well known and non-trivial theorem, and while not difficult we will not present any here since we are not interested in the mathematical foundations of signal processing. Instead, we will just use it for our purposes. This theorem has a one-dimensional analogue[2]:

$$\begin{cases} s(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx & \text{(CFT)} \\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(k) e^{2\pi i k x} dk & \text{(ICFT)} \end{cases}$$

We can now say that the image[3] f(**r**) and signal s(**k**) are related via a CFT (and ICFT):

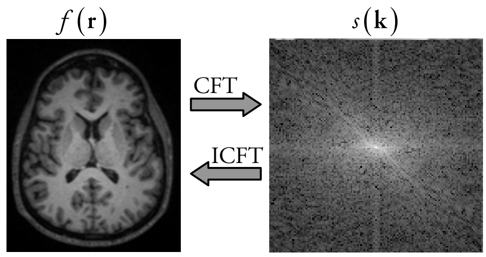


Image space k-space

Note: since the CFT and ICFT definitions are so similar, authors often switch between the CFT and ICFT, and might call our ICFT a CFT. There really is very little difference apart from a scaling factor (1/2π) and flipping the resulting transform (which has –k instead of k in the exponent). Our particular choice of names stems from trying to mimic as closely as possible the way the popular academic programming language MATLAB handles Fourier transforms.

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[2] The signs and 1/2π factors differ between different textbooks and papers. This is a consequence of the Fourier transform theorem: one only needs the signs in the exponential in the CFT and ICFT to be opposite. Similarly, one needs the factors of the integrals to equal $\frac{1}{2\pi}$, so in some books both the CFT and ICFT have a $\frac{1}{\sqrt{2\pi}}$ factor in front of them, and in some books the CFT has the $\frac{1}{2\pi}$ factor in front of it.

[3] Or, to be exact, of the image weighted by the receiver sensitivity pattern and, later on, by the relaxation.