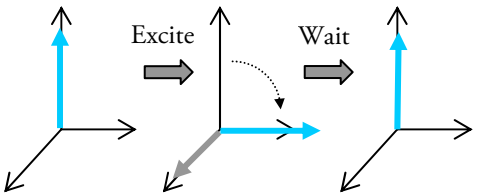
III
Spins and their Thermodynamics

On the menu:

1. Relaxation & thermal equilibrium
2. Relaxation: phenomenology (T_1, T_2)
3. Relaxation: microscopic description
4. T_2* dephasing
5. Measuring T_1, T_2 and T_2*

We've remarked:



The three questions of relaxation:

1. **What** is the equilibrium state?
2. **How** do the spins return to it? (macro)
3. **Why** do the spins return to it? (micro)

1. Thermal Equilibrium

1.1 Boltzmann's Principle

The whole of statistical physics rests on the following Boltzmann hypothesis: at thermal equilibrium, the probability of the system being in a state with energy E is:

$$\Pr(E) = \frac{1}{Z} e^{-E/kT}$$

where Z is a constant number independent of the energy or kT. If you have N states with energies $E_1,\ldots,E_N$, then the probability of being in state i is:

$$\Pr(i) = \frac{1}{Z} e^{-E_i/kT}$$

We want this to be an actual probability, i.e.,

1. Be between 0 and 1.
2. Sum to 1.

So:

$$\Pr(1) + \Pr(2) + \ldots + \Pr(N) = 1$$

or

$$\frac{1}{Z} e^{-E_1/kT} + \ldots + \frac{1}{Z} e^{-E_N/kT} = 1$$

This allows us to solve for Z:

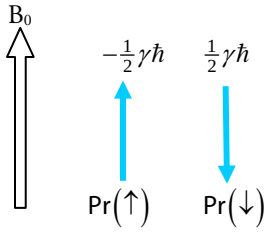
$$Z = e^{-E_1/kT} + \ldots + e^{-E_n/kT}$$

1.2 Thermal Equilibrium of Spin-1/2

This is the only point in our lectures where we'll use the quantum-mechanical nature of spin. Quantum mechanics tells us a spin-1/2 in a field B_0 has two energy levels:

$$E_\uparrow = -\tfrac{1}{2}\hbar\gamma B_0 \quad \text{(aligned along } B_0\text{)}$$
$$E_\downarrow = \tfrac{1}{2}\hbar\gamma B_0 \quad \text{(aligned against } B_0\text{)}$$

This does NOT mean a spin must point along B_0 in thermal equilibrium. It only means that *statistically*.



We now ask: *what is the average magnetic moment of (one) spin ½ at equilibrium*?

$$< m_x > = < m_y > = 0$$
$$< m_z > = \tfrac{1}{2}\gamma\hbar \Pr(\uparrow) - \tfrac{1}{2}\gamma\hbar \Pr(\downarrow)$$

If we knew that, then for N non-interacting spins (and spins are non interacting in our world, let me assure you) at equilibrium,

$$\mathbf{M} = N < m_z > \hat{\mathbf{z}}$$

which would be our equilibrium magnetic moment. So we really just need to compute $<m_z>$.

Now,

$$Z = e^{\frac{\hbar\gamma B_0}{2kT}} + e^{-\frac{\hbar\gamma B_0}{2kT}}$$

This unpleasant expressions can be simplified considerably if we remember that, for small a (<<1):

$$e^a \approx 1 + a$$
$$(\text{e.g. } e^{-0.01} \approx 0.99)$$

In our case, at room temperature (homework!),

$$\frac{\hbar\gamma B_0}{kT} << 1$$

so we can simplify:

$$Z \approx \left(1 + \frac{\hbar\gamma B_0}{2kT}\right) + \left(1 - \frac{\hbar\gamma B_0}{2kT}\right) = 2$$

Hence,

$$\Pr(\uparrow) = \frac{e^{\frac{\gamma\hbar B_0}{2kT}}}{Z} \approx \frac{1}{2}\left(1 + \frac{\gamma\hbar B_0}{2kT}\right)$$
$$\Pr(\downarrow) = \frac{e^{-\frac{\gamma\hbar B_0}{2kT}}}{Z} \approx \frac{1}{2}\left(1 - \frac{\gamma\hbar B_0}{2kT}\right)$$

So, what is the average magnetic moment of a spin ½ at equilibrium?

$$< m_z > = \frac{1}{2}\gamma\hbar\Pr(\uparrow) - \frac{1}{2}\gamma\hbar\Pr(\downarrow)$$
$$= \frac{1}{4}\gamma\hbar\left(1 + \frac{\gamma\hbar B_0}{2kT}\right) - \frac{1}{4}\gamma\hbar\left(1 - \frac{\gamma\hbar B_0}{2kT}\right)$$
$$= \frac{(\gamma\hbar)^2 B_0}{4kT}$$

and, for N spins,

$$\boxed{M_0 = \frac{N(\gamma\hbar)^2 B_0}{4kT}}$$

Equilibrium magnetic moment

You will show in the homework that, for a 1 cm^3 voxel with just water (tissue has ~ 70-80% water, so that's reasonable),

$$M_0 \approx 3.7 \times 10^{-8} \frac{\text{Joule}}{\text{Tesla}}$$

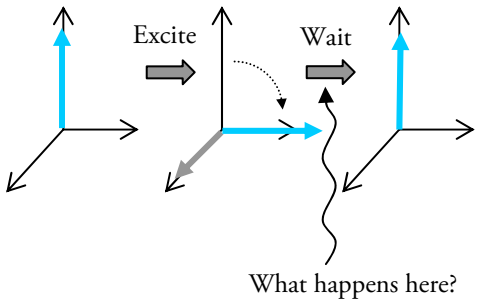
Note this has units of energy per unit field. You can think of it as the amount of energy you create when you put the water in a field of such and such Tesla. It's quite small!

Some notes:

1. M_0 is small.
2. The signal we measure will be proportional to M_0.
3. Hydrogen nuclei (water!) have the largest γ and hence the largest signals. How lucky we are that nature somehow turned out that way!
4. Proportional to B_0: want a larger signal? Go to higher fields.
5. Careful when using this formula. N is basically determined by the natural abundance, etc. For example, if you have 100 water molecules then N should be 200, because you have 2 Hydrogen atoms per molecule. Actually it should be more like 200*0.99985 if you'd like to take into account Hydrogen's natural abundance (99.985%).

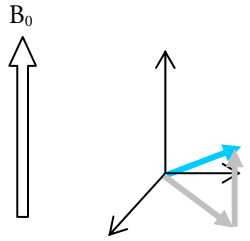
2. Phenomenology

2.1 T_1 and T_2



It turns out that the component of magnetization **perpendicular** to B_0 (z-axis) and the component **parallel** to it "relax" differently.

Every magnetization vector **M** can be decomposed to a parallel component (to B_0, here parallel to z) and a perpendicular one:



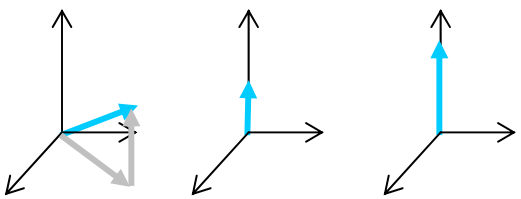
It turns out:

1. The transverse magnetization gets "eaten up" and eventually disappears with a time constant T_2. Usually T_2 ~ tens of ms in tissue.
2. The longitudinal magnetization gets "built up" back to its equilibrium value, with a time constant T_1 which is always larger (but not always by much!) than T_2.
3. Remember, $T_1 \geq T_2$ **always**. This means the transverse magnetization gets "eaten up" faster than the longitudinal magnetization gets "built up".

Here are some typical values (taken from Haacke 1999):

	T_1 (ms)	T_2 (ms)
GM	950	100
WM	600	80
Muscle	900	50
CSF	4500	2200
Fat	250	60
Blood	1200	100 (venous)

Here's a "movie" of what'll happen in a tissue with $T_1 >> T_2$ such as GM or WM (so transverse relaxation occurs "before" longitudinal):



Initially … T_2 relaxation T_1 relaxation

2.2 Bloch Equations

Bloch found that the equations we've found last chapter,

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B},$$

in components:

$$\frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y$$
$$\frac{dM_y}{dt} = \gamma M_z B_x - \gamma M_x B_z$$
$$\frac{dM_z}{dt} = \gamma M_x B_y - \gamma M_y B_x$$

can be modified to describe the relaxation phenomenologically:

$$\frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y - \frac{M_x}{T_2}$$
$$\frac{dM_y}{dt} = \gamma M_z B_x - \gamma M_x B_z - \frac{M_y}{T_2}$$
$$\frac{dM_z}{dt} = \gamma M_x B_y - \gamma M_y B_x - \frac{M_z - M_0}{T_1}$$

Notes:

1. M_0 is the thermal equilibrium magnetization, computed last section.
2. These equations hold in the rotating frame as well.
3. Note how M_x, M_y has new terms with T_2 and M_z has a new term with T_1, expressing the fact the longitudinal and transverse relaxations occur with different time rates.

What can we make of these? Let's look at a simple case:

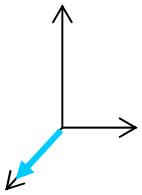
1. We've just excited our spins.
2. We're on resonance (no offset, B_z=0).
3. No irradiation (B_x=B_y=0).

Our equations become:

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2}$$
$$\frac{dM_y}{dt} = -\frac{M_y}{T_2}$$
$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

with the initial condition

$$\mathbf{M}(t=0) = M_0\hat{\mathbf{x}}$$



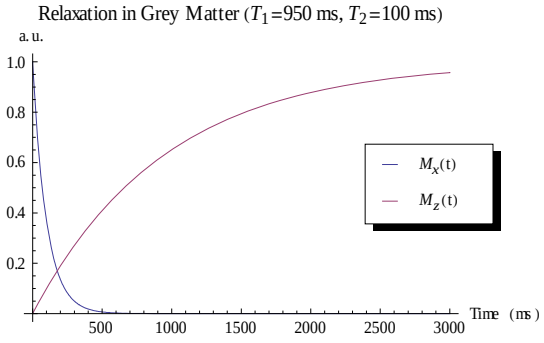
We can solve these equations. First, M_y=0 initially and it will stay so, because M_y=0 solves the 2nd equation. How about M_x? This is a typical differential equation y'=ay with a solution $y(t)=y_0e^{ax}$. Hence, the solution is:

$$M_x(t) = M_0 e^{-t/T_2}$$

M_z is slightly trickier to solve for, and I'll leave that as a homework problem. The solution is:

$$M_z(t) = M_0\left(1 - e^{-t/T_1}\right)$$

Plotted (for M_0=1):



In general:

1. Your spins rotate with the applied external fields.
2. Meanwhile, T_2 and T_1 act in the background, causing the magnetization to slowly return to its equilibrium.

3. A Look Under the Hood

3.1 Microscopic Mechanism

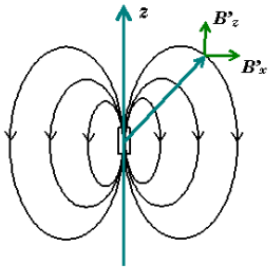
What causes relaxation? We have three mechanisms to account for:

$$\frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y - \frac{M_x}{T_2}$$
$$\frac{dM_y}{dt} = \gamma M_z B_x - \gamma M_x B_z - \frac{M_y}{T_2}$$
$$\frac{dM_z}{dt} = \gamma M_x B_y - \gamma M_y B_x - \frac{M_z - M_0}{T_1}$$

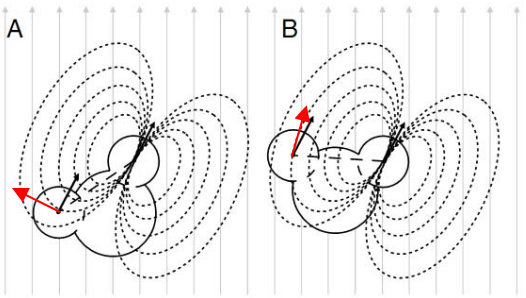
1. Red: why do M_x, M_y decay?
2. Green: why does M_z changes with a different time constant, T_1?
3. Blue: why does buildup occur?

The buildup part (#3) is difficult to explain and we'll have to leave it at that. The decay of **M**, with different transverse & longitudinal times, will be explained next.

The nuclear magnetic moments each create their own fields:

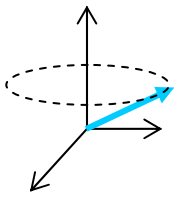


Now think about the following experiment: take two spins, fix one and move the other one about. Just by virtue of moving, the field "felt" by the spin changes:

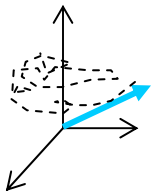


(Red: magnetic field)

Since all water molecules keep tumbling and moving around (a.k.a. brownian motion), each sees the main field + a fluctuation field created by the other spins. These fluctuating fields are what cause relaxation. You can think of the spin of a **single** molecule as "tumbling" on the surface a sphere:



Without fluctuating fields: precession



With fluctuating fields: precession + erratic "jumps" (not drawn to scale, etc.)

with the end result being the total magnetic moment decays back to equilibrium.

The fluctuating fields $\mathbf{B}_D$ felt by a spin can also be composed into transverse & longitudinal components:

$$\mathbf{B}_D(t) = \mathbf{B}_{D,\perp}(t) + \mathbf{B}_{D,\parallel}(t)$$

The longitudinal fluctuating field causes transverse relaxation and the transverse fluctuating field causes the longitudinal relaxation.

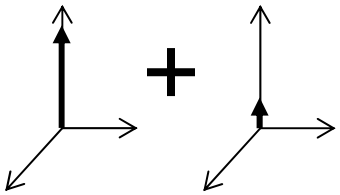
3.2 Transverse Relaxation

Why does the transverse magnetization get "eaten up"? Let's work in the lab frame. Imagine first no fluctuating fields. A bunch of spins in an isochromat would all rotate with the same Larmor frequency, $\omega_0=\gamma B_0$.

Now imagine each spin feels a fluctuating field along the z-direction, so its precession frequency also becomes time dependent:

$$\omega(t)=\gamma(B_0+B_{D,\parallel}(t))$$

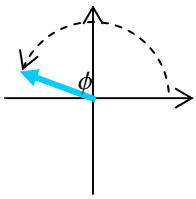
Total field =



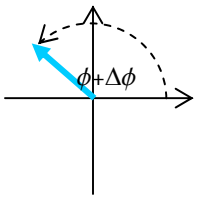
Main B_0 field (large, ~ Tesla)

Smaller dipolar fluctuating longitudinal field (~

Imagine exciting a spin onto the xy plane. Without the fluctuating field, it would just execute precession and make a phase $\phi=\gamma B_0$. With the fluctuating field along z the precessing frequency fluctuates as well, with the end result being a slightly different precessing frequency at the end, $\phi+\Delta\phi$, where $\Delta\phi$ depends on the exact nature of the fluctuations (imagine turning a wheel with a shaking hand):

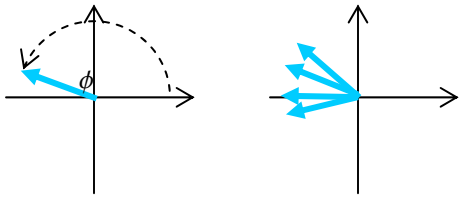


No fluctuations ("Firm hand")



With fluctuations ("Shaky hand")
Note here $\Delta\phi<0$

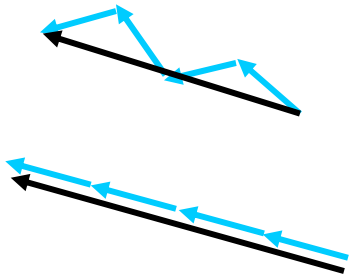
Now imagine a number of spins. In the absence of fluctuations they would all make the same angle. In the presence of fluctuations, they would fan out (remember, each spin feels a different fluctuation):



Many spins,
no fluctuations.
(microscopic view)

Many spins,
fluctuations.
(microscopic view)

This is what happens microscopically. Now, the **macroscopic** magnetization is the (vector) sum of the microscopic magnetization. What happens when you sum vectors that don't point in the same direction? They (partially) cancel out. Example:



Top: summing 4 vectors not pointing in the same direction. Bottom: all 4 vectors point in the same direction. In both cases, the "mini-vectors" (blue) all have the same size. You can now see why the magnetization in the plane decays:

> The fluctuating z-field causes the spins to spread out (**dephase**), and hence add up destructively, leading to a decay of the macroscopic magnetization vector, **M**.

How fast does **M** decay – what determines T_2? Quite simply: the rate of fluctuations. Fast fluctuations will result in lesser dephasing and hence slower decay.

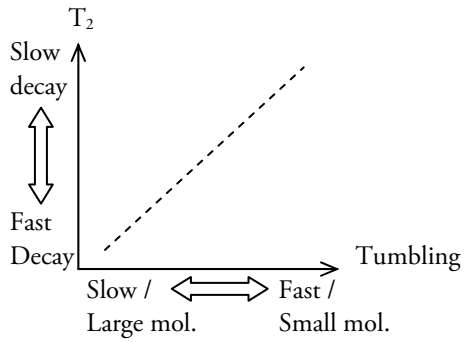
An analogy might help see this: think of diffusion. Molecules randomly change their direction upon colliding with each other. It should be intuitively apparent that, the lower the concentration of your sample, the larger the diffusion. Here the story is the same: you can think of the spin as "diffusing" under the action of the fluctuating field – slower fluctuations mean "fewer collisions" and hence a "less dense" environment, leading to greater "diffusion" (dephasing, in our case).

This directly relates to molecule sizes, because:

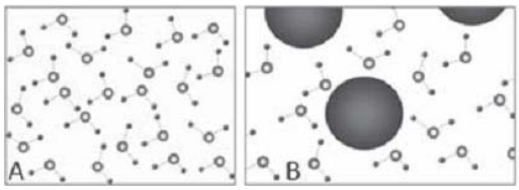
Large molecules
- → Tumble slowly
- → Slow fluctuations
- → Small T_2 (fast decay)

Small molecules
- → Tumble fast
- → Fast fluctuations
- → Small T_2 (slow decay)

Hence we can draw this graph:

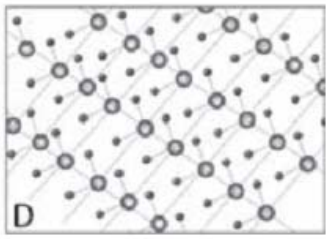


In tissue, water can be free (A) or in the vicinity of large macromolecules (B), which slow it down and lengthens its T_2:



Fast, long T_2 Slower, shorter T_2

In a solid, where there is almost no motion, T_2 is extremely short:



This is why, e.g., bone cannot be imaged (it's a solid with T_2 ~ 0.01 ms - it returns to equilibrium too fast for us to measure it!).

3.3 Longitudinal Relaxation

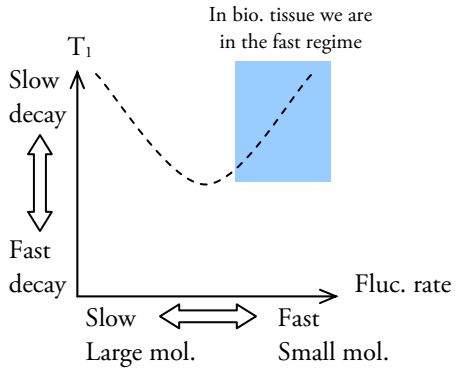
The x & y components of the fluctuating fields cause longitudinal relaxation. This can be easily understood if you can think of these fields as tiny "RF pulses" that tilt the magnetization.

Remember the idea: for an RF pulse to be successful, it needs to be **on resonance**. The rate of fluctuations determines whether the RF is on resonance or not: when the field fluctuates at the same frequency as the spin,

$$\omega_{\text{fluctuations}} = \omega_0 = \gamma B_0$$

the tiny "RF pulses" tilt the magnetization back to equilibrium much more efficiently, hence making T_1 shorter. Too fast or too slow – and you won't be on resonance anymore, diminishing the relaxation.

As before, we can draw:



In solids, for example, we saw T_2 is very short, but T_1 will be very long.

As an interesting application, let's apply our microscopic insight to understand why T_2 and T_1 values in tumors are larger than in regular tissue. Cancer is usually edematous: cells swell with water, making the macromolecule concentration lower, making the water molecules tumble faster, increasing T_2 and T_1.

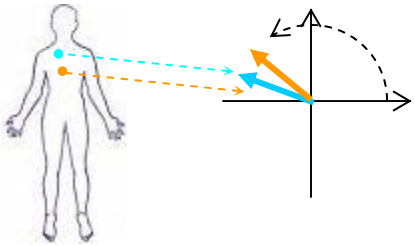
Cancer → Swelling
→ Lower macromol. concentration
→ Faster tumbling of water molecules
→ Larger T_1, T_2 (slower decay)

4. T_2^* Dephasing

An additional source that causes the macroscopic magnetization **M** to diminish in size is known as T_2' dephasing. It is caused by **spatial** field variations along the z-direction:

$$B_0\hat{\mathbf{z}} \longrightarrow \left[B_0 + \Delta B_0(z)\right]\hat{\mathbf{z}}$$

This leads to dephasing, this time between spins at different locations (adjacent or not):



The spins in the body are excited and left to evolve. Tiny variations in the main field mean spins at different positions precess with slightly different angular velocities and eventually "get out of sync" (i.e. dephase)

Contrast this with T_2, which is caused by **temporal** field variations:

Field variations cause relaxation:
Spatial, constant → T_2'
Temporal fluctuations → T_2

What causes these field non-homogeneities?

1. Hardware imperfections of B_0.
2. Susceptibility artifacts: the external field $\mathbf{B}_0 = B_0\mathbf{z}$ affects the electron spins in different parts of the body slightly differently, depending on the properties

of the tissue. This creates small variations in the z-field B_0.

In the Bloch equations, both T_2 and T_2' play a smilar role. For example, for the x component, instead of

$$\frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y - \frac{M_x}{T_2}$$

Decay due to field fluctuations

we'd have:

$$\frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y - \frac{M_x}{T_2} - \frac{M_x}{T_2'}$$

Decay due to nonhomogeneity

In general, one introduces a new decay quantity, T_2*, that combines the effects of both T_2 and T_2':

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

This way we can rewrite the bloch equations – e.g., for the x-component:

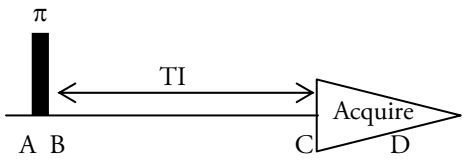
$$\frac{dM_x}{dt} = \gamma M_y B_z - \gamma M_z B_y - \frac{M_x}{T_2^*}$$

Takes into account both field fluctuations (temporal) and inhomogeneities (spatial)

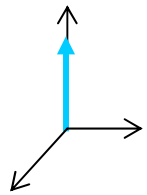
5. Measuring T_2, T_1, and T_2*

5.1 T_1 - Inversion Recovery

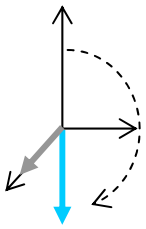
To measure T_1 of water, consider the following experiment:



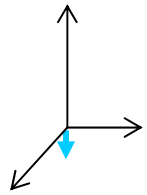
Let's go through what happens to the magnetization at each of the points outlined above.
A.) The magnetization is at thermal equilibrium,
B.) A hard π-pulse is used to flip the magnetization onto the –z axis.
C.) We wait a time TI. Longitudinal relaxation kicks into effect.
D.) We excite the spin onto the xy-plane and measure. For the sake of simplicity, we can take the magnitude of the initial signal.



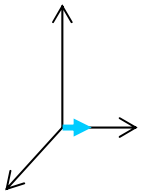
(A.) Thermal eq.



(B.) After π-pulse



(C.) After time TI



(D.) Precession (measure)

The amount of decay depends on the time TI we'd wait. We can solve the Bloch equations:

$$\begin{cases} \dfrac{dM_z}{dt} = -\dfrac{M_z - M_0}{T_1} \\ \text{initial condition: } M_z(t=0) = -M_0 \end{cases}$$

To solve, substitute:

$$Y = M_z - M_0$$
$$\frac{dY}{dt} = \frac{dM_z}{dt}$$
$$Y(0) = M_z(0) - M_0 = -2M_0$$

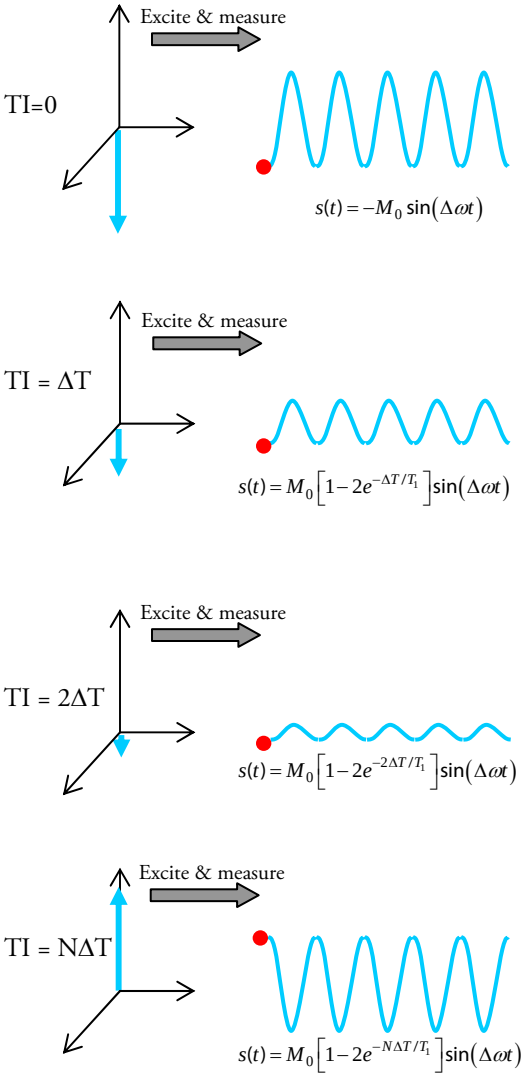
so:

$$\frac{dY}{dt} = -\frac{Y}{T_1} \quad \Rightarrow \quad Y(t) = -2M_0 e^{-t/T_1}$$

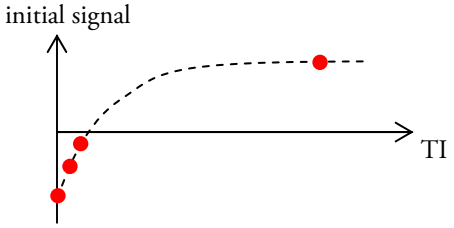
Substituting back Y in terms of M_z, we recover the solution:

$$M_z(t) = M_0\left[1 - 2e^{-t/T_1}\right]$$

This will determine the amplitude of the signal after waiting a time TI. We can imagine a set of experiments done with different TIs. In each experiment, the maximal value of the signal is taken:



Next, you can imagine taking the initial amplitude of each decay and graphing it. You will then be able to directly observe the decay of M_z and deduce T_1:



By fitting this decay curve to

$$M_z(t) = M_0\left[1 - 2e^{-t/T_1}\right]$$

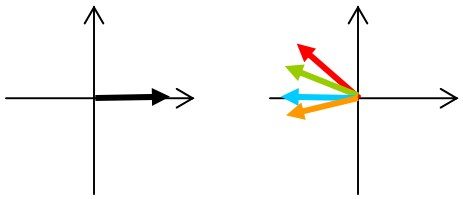
you can find T_1. This is called an **inversion recovery** (IR) experiment.

5.2 T_2 – Spin Echo Experiment

Imagine having a sample with spins having different offsets. This can come about in several ways, and here are two:

1. Non-homogeneity (T_2') of B_0.
2. A gradient is turned on.

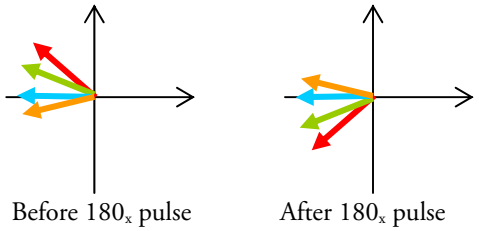
Once you excite the spins from thermal equilibrium, they begin precessing at different rates, and eventually "spread out" in the xy-plane. This means that, if you were to acquire their signal, it would slowly die out because the spins would end up pointing in all sorts of directions and add up destructively (remember, the signal is a vector sum of the spins in the xy-plane):



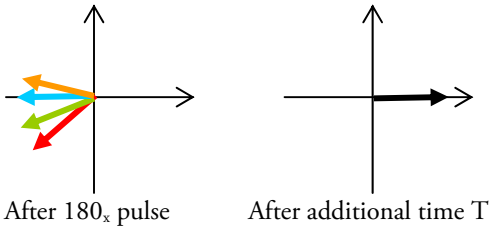
Following excitation, all spins point in the same direction

However, since each spin precesses at a different rate (= different colors), they end up dephasing

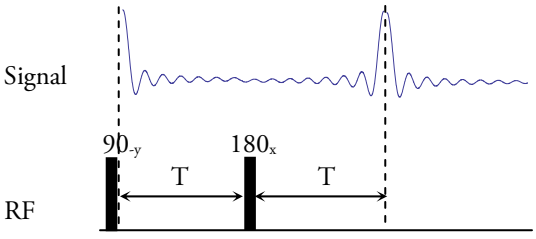
What would happen if we were to apply a 180 pulse along, say, the x-axis, after a time T? The pulse would invert our spins:



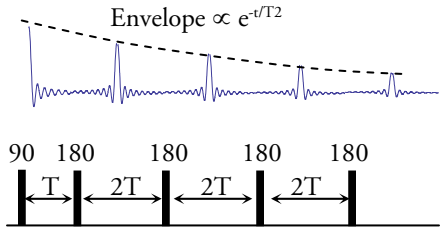
However, note the interesting part: if we were to wait an additional time T, the spins would end up re-aligning along the x-axis:



The reason for this can be understood by thinking of a particular spin: suppose a particular spin acquired some phase ϕ just prior to the 180 pulse. After the pulse, its phase would be $-\phi$. After a time T its phase would increase by ϕ once again, so its phase at the end would be $(-\phi)+\phi = 0$, i.e., it's back at the x-axis. If we'd continue acquiring throughout this experiment, we'd end up seeing the signal revive back again. This is called a **spin echo**. In terms of pulse sequences:



What would happen if we were to give successive 180_x pulses, spaced 2T apart? One might initially think this pattern would repeat itself indefinitely, since the spins would dephase, get flipped (by the 180), rephase, dephase again, get flipped (by the 180), rephase, dephase, ... ad infinitum; in effect, there is relaxation that needs to be taken into account. But what relaxation? This spins are in the plane, but it's **not** T_2^*; rather, it is T_2. Because the 180 pulse refocuses spins with different precession frequencies, there are no T_2' effects in the overall decay. Only the "true microscopic decay", T_2, plays a role here:



The decay after the excitation is determined by T_2^* (by both microscopic field fluctuations and field non-homogeneities), but the overall decay of the echoes is determined by T_2 alone. This furnishes us with a method of measuring the "true" T_2 microscopic decay of a sample.