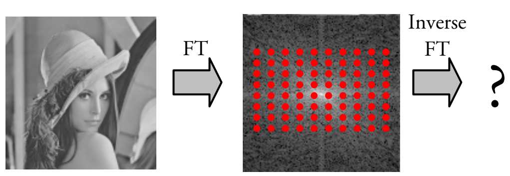
VII
Resolution

Lecture notes by Assaf Tal

1. The Effects of a Finite Sampling Grid

The fact that the FT of the image is sampled at only a discrete number of points means the reconstruction of the image will not be perfect:

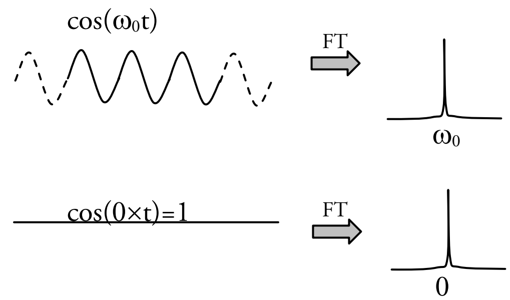


We will now argue two things:

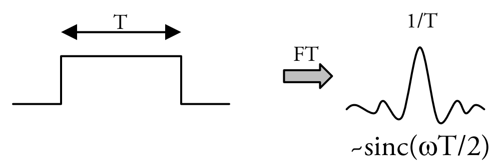
1. The discreteness of the grid will lead to the main image being duplicated infinitely in all directions. The "original", "main" image and its duplicates will be spaced apart in inverse proportion to the spacing in k-space: the denser the grid, the more spaced-apart the duplicates. If the points in k-space will be too far apart, the duplicates will end up overlapping with the "main" image and botch it up. This is called **aliasing**.
2. The finite extent of the grid will cause a basic blurring in the image, which will be greater the smaller the grid.

1.1 Blurring

We now turn to hand-wave our way through these two arguments. To make things easier to visualize, we'll deal with the 1D case, and finally remark something about the 2D case. The Fourier transform of the function cos(ωt) is a very sharp peak (a "delta function", to be mathematically precise, but never mind the exact name):



We've seen, however, that:

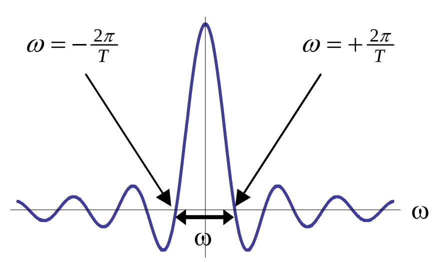


i.e., by "cutting off" the constant function cos(0×t)=1 at the edges, effectively turning it into a "box" of width T, we end up widening the sharp peak and turning it into a sinc function.

The width of this sinc function is inversely proportional to the function's width, T. Reasoning: the Fourier transform of a "box" of width T is (up to constant factors):

$$\hat{f}(\omega) = \mathrm{sinc}\left(\frac{\omega T}{2}\right) = \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} .$$

Note that $\hat{f}(\omega)$ reaches 0 first when $\frac{\omega T}{2} = \pm\pi$, i.e., when $\omega = \pm\frac{2\pi}{T}$, at which point the nominator equals zero: $\sin\left(\frac{\omega T}{2}\right) = 0$. The function, when plotted, looks like this:



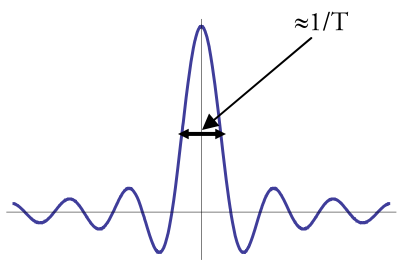
So the distance between the first two zeros is

$$\Delta\omega = \frac{2\pi}{T} - \left(-\frac{2\pi}{T}\right) = \frac{4\pi}{T}$$

and, in terms of frequencies (not angular frequencies):

$$\Delta\nu = \frac{\Delta\omega}{2\pi} = \frac{2}{T}$$

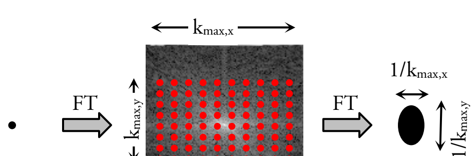
The "width" of the sinc is a somewhat loose concept, but you can see that it is equal to about 1/T:



The conclusion is:

> Truncating the sampled function at a width T, causes sharp features to widen ("blur") to a width 1/T.

In terms of k-space, a sharp point in the image at f(x,y) will get blurred by an amount proportional to $k_{max,x}$ in the x-direction and $k_{max,y}$ in the y-direction:



Left: initial object to be imaged is sharp. Middle: sampled k-space (schematic drawing). Right: Fourier-transforming the sampled k-space data points results in widened peak.

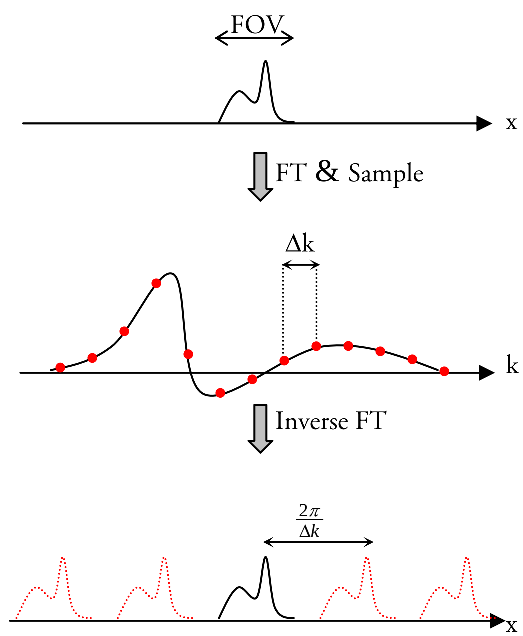
Objects which change on a coarser scale compared to $1/k_{max,x}$ along the x-direction and $1/k_{max,y}$ along the y-direction *do not get blurred* to a good approximation. Thus, the finite extent of the sampling defines an **inherent voxel[1] size**: a scale beneath which things get blurred and cannot be detected.

1.2 Aliasing

We next turn to the effect of the "graininess" of the grid. My claim is:

> Sampling a function f(k) at intervals of Δk will create an infinite number of copies of it, spaced $\frac{2\pi}{\Delta k}$ apart.

Graphically, in 1D:



This in general is not a problem, unless the clones overlap the original function and interfere with it. This sort of problem is called

[1] A voxel means a "volumetric pixel", i.e. a "pixel in 3D". It is a term commonly used in medical physics.

aliasing. If we denote by FOV[2] the size of the object being imaged, then aliasing will occur whenever:

$$\mathrm{FOV} \geq \frac{2\pi}{\Delta k}$$
(for aliasing)

Your goal when sampling is usually to reduce Δk enough for this <u>not</u> to happen.

Let me try and justify (in a most hand-wavy manner) the appearance of the copies. Consider the function cos(ωt) again, sampled at time intervals of Δt. Now, given those discrete samples, can you guess ω? The answer is: *not uniquely*. That is, there will be an infinite number of ω's that will fit the sampled points, and those will be:

$$\omega + \frac{2\pi n}{\Delta t}, \qquad n \text{ any integer}$$

This is the source of the copies. For example, sampling cos(ωt) at intervals of $\Delta t = \frac{2\pi}{\omega}$ will yield the sampled points: $s_n = \cos(\omega n \Delta t) = \cos(2\pi n) = 1$, i.e. a stream of 1s: {1, 1, 1, 1, 1, ... } (because $\cos(2\pi n) = 1$ for all n). Note that there are many periodic functions that fit this constant sequence:

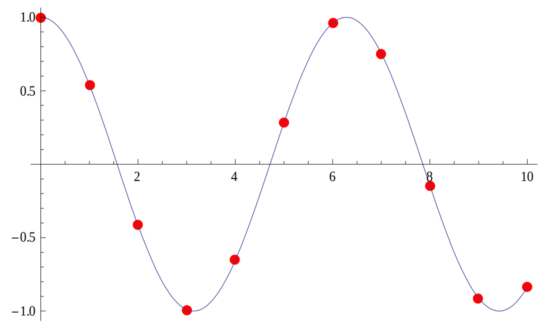
$$f_n(t) = \cos\left(\left(\omega + \tfrac{2\pi n}{\Delta t}\right)t\right), \quad \text{n any integer},$$

because they all give a sequence of 1s when evaluated on the sampled grid (at t=kΔt):

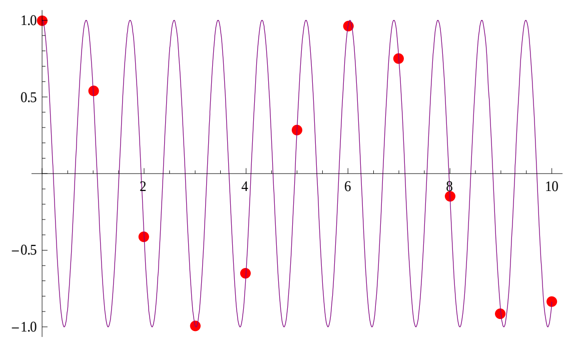
$$\begin{aligned} f_n(k\Delta t) &= \cos\left(\left(\omega + \tfrac{2\pi n}{\Delta t}\right)k\Delta t\right) \\ &= \cos(\omega k \Delta t + 2\pi n k) \qquad \left(\text{use } \Delta t = \tfrac{2\pi}{\omega}\right) \\ &= \cos(2\pi k) = 1 \end{aligned}$$

[2] FOV stands for "Field of View", used to describe the size of the object being imaged. For 2D and 3D objects, the FOV can be different along different axes.

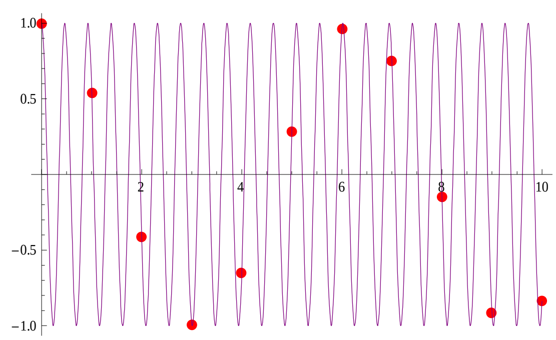
Here's a graphical example: let's take the function f(t) = cos(t), and sample it at intervals Δt = 1:



Note that the function $\cos\left(\left(1 + \frac{2\pi}{\Delta t}\right)t\right)$ also fits these dots:

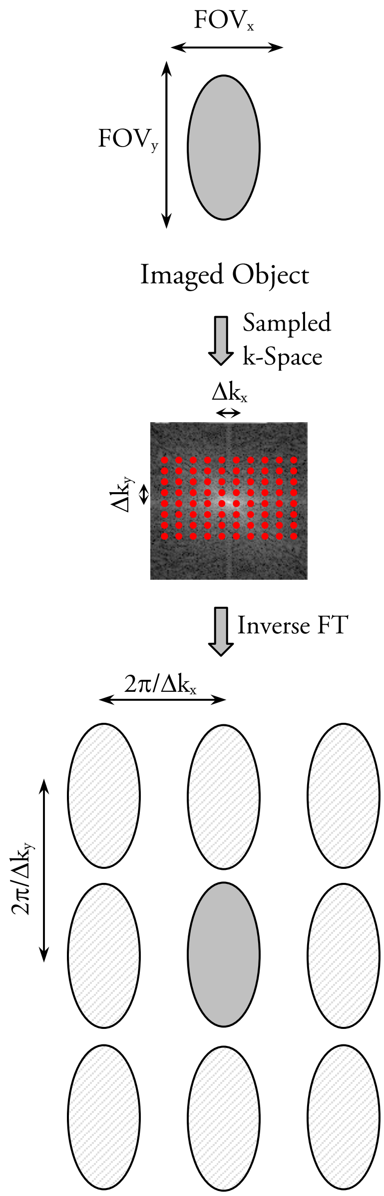


as does the function $\cos\left(\left(1 + 2 \times \frac{2\pi}{\Delta t}\right)t\right)$:



et cetera. The FT of the sampled version (the red dots), being "a black box that detects periodicities", will produce peaks at all possible frequencies that fit the red dots.

For 2D objects, this aliasing is manifested as copies of the object being imaged along both the x & y axes:



As with the 1D case, to avoid aliasing you'd have to set (in 2D):

$$\frac{2\pi}{\Delta k_x} \geq \text{FOV}_x, \qquad \frac{2\pi}{\Delta k_y} \geq \text{FOV}_y$$

If your image is 3D, an analogous condition in 3D would apply (assuming you sample that direction and not selectively excite!).

2. Relation to Imaging Parameters

The concept of FOV, voxel size and resolution are intimately related. In a 2D image, the number of voxels in an image along the x and y axes is:

$$N_x = \frac{\text{FOV}_x}{\Delta x}, \qquad N_y = \frac{\text{FOV}_y}{\Delta y}$$

where Δx and Δy are the voxel sizes. Assuming that (e.g., for the x-direction) we select Δk_x to equal $1/\text{FOV}_x$, we have:

$$\Delta x = \frac{1}{k_{\max,x}}, \qquad \text{FOV}_x = \frac{1}{\Delta k_x}$$

and so:

$$N_x = \frac{FOV_x}{\Delta x} = \frac{k_{\max,x}}{\Delta k_x}$$

This relates the k-space quantities to the resolution of the image.