

Geometrical optics

Textbook: Born and Wolf (chapters 3-5)

Overview:

Electromagnetic plane waves (from Maxwell's equations).

The Eikonal equation and its derivation (optics at zero wavelength).

Refraction ; Snell's law (reflection) ; Total internal reflection ; The prism ; Dispersion ; The thin lens

Imaging as a projective transformation.

Optical systems and the ABCD matrix.

The primary optical aberrations ; Chromatic aberration.

1. From Maxwell's equation through the wave equation to the eikonal equation – up to the derivation of Snell's law.

Start with Maxwell's equations in a dielectric medium (no currents, no charges)

$$\begin{aligned}\nabla_x H - \frac{1}{c} \frac{\partial D}{\partial t} &= 0 & \nabla \cdot D &= 0 \\ \nabla_x E - \frac{1}{c} \frac{\partial B}{\partial t} &= 0 & \nabla \cdot B &= 0\end{aligned}$$

Along with the material equations:

$$\begin{aligned}D &= \varepsilon E \\ B &= \mu H\end{aligned}$$

Substituting the equation for B into the second Maxwell's equation and taking a curl operation we get:

$$\nabla_x \left(\frac{1}{\mu} \nabla_x E \right) - \frac{1}{c} \nabla_x \frac{\partial H}{\partial t} = 0$$

From which we get, by applying the second Maxwell equation:

$$\nabla_x \left(\frac{1}{\mu} \nabla_x E \right) - \frac{\varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

multiplying by μ and using $\nabla_x UV = U \nabla_x V + \nabla U x V$ we get:

$$\nabla_x \nabla_x E + \mu \nabla \left(\frac{1}{\mu} \right)_x (\nabla_x E) - \frac{\mu \varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

and using the identity $\nabla_x \nabla = \nabla^2 + \nabla(\nabla \cdot)$ we get:

$$\nabla^2 E + \nabla(\nabla \cdot E) + \mu \nabla \left(\frac{1}{\mu} \right)_x (\nabla_x E) - \frac{\mu \varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

Since $\nabla \cdot D = 0$ (from Maxwell's equations), we get:

$$\nabla \cdot (\varepsilon E) = 0 \Rightarrow \nabla \cdot E = -E \cdot \frac{\nabla \varepsilon}{\varepsilon}$$

Finally, we can get the end result:

$$\nabla^2 E - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 E}{\partial t^2} + (\nabla \ln \mu)_x (\nabla_x E) + \nabla(E \cdot \nabla \ln \varepsilon) = 0$$

in non-magnetic media $\mu=1$, and ϵ varies between 1 and ~ 20 at optical frequencies, so the last term is generally slowly varying. For homogeneous media we get:

$$\nabla^2 E - \frac{\epsilon\mu}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

From which we get the speed of light in a medium (phase velocity) $v = c/\sqrt{\mu\epsilon}$, thus $\sqrt{\mu\epsilon}$ is defined as the refractive index of the material

Harmonic plane waves and the Eikonal approximation

Clearly, a plane wave of the form:

$$E(\mathbf{r}, t) = E_0 e^{i(kx - \omega t)}$$

is a solution of the wave equation if $\omega/k=v$. (similarly for H). This is a plane wave. We will consider other solutions to propagation in free space later.

Let us now derive the wave equation assuming that the wavelength is small. This should provide a decent approximation as long as we are many wavelengths away from sources or boundaries.

For this we can assume an Ansatz of the form:

$$E(\mathbf{r}, t) = e_0(\mathbf{r}) e^{ikS(\mathbf{r})} e^{i\omega t}$$
$$H(\mathbf{r}, t) = h_0(\mathbf{r}) e^{ikS(\mathbf{r})} e^{i\omega t}$$

where S is a scalar function, and write the wave equation keeping only the highest order terms in k (assuming λ is small relative to all gradients of e,h,S).

From this we get:

$$\mu \frac{\partial H(\mathbf{r}, t)}{\partial t} = \nabla \times E(\mathbf{r}, t) = \nabla \times e_0(\mathbf{r}) e^{ikS(\mathbf{r})} e^{i\omega t} + ik \nabla S \times E(\mathbf{r}, t) \approx$$
$$\approx ik \nabla S \times E(\mathbf{r}, t)$$

$$\varepsilon \frac{\partial E(\mathbf{r}, t)}{\partial t} = \nabla \times H(\mathbf{r}, t) = \nabla \times h_0(\mathbf{r}) e^{ikS(\mathbf{r})} e^{i\omega t} + ik \nabla S \times H(\mathbf{r}, t) \approx$$
$$\approx ik \nabla S \times H(\mathbf{r}, t)$$

The highest order term in k leaves just the gradient of S, and the time harmonic term (applying the time derivative twice), gives:

$$\varepsilon \mu \omega^2 e_0 = k^2 (\nabla S)^2 e_0$$
$$\left[n^2 - (\nabla S)^2 \right] e_0 = 0$$

From Maxwell's equations it is simple to write the Poynting vector, which is proportional to $E \times H$. Since from the above:

$$i\omega H(\mathbf{r}, t) \approx ik \nabla S E(\mathbf{r}, t)$$

this can be shown to be equal to:

$$\frac{2c}{n^2} \langle w_e \rangle \nabla S$$

where $\langle w_e \rangle$ is the time averaged energy in the electric field. The poynting vector is thus oriented along $\text{grad}(S)$.

Defining a ray as a path $r(s)$ propagating along the poynting vector we can write:

$$\frac{dr}{ds} = \frac{\nabla S}{n}$$

From which we can derive:

$$\frac{d}{ds} n \frac{dr}{ds} = \frac{d\nabla S}{ds} = \frac{dr}{ds} \nabla(\nabla S) = \frac{1}{n} \nabla S \cdot \nabla(\nabla S) = \frac{1}{2n} \nabla[(\nabla S)^2] = \frac{1}{2n} \nabla(n^2)$$

or

$$\frac{d}{ds} n \frac{dr}{ds} = \nabla n$$

In particular, for constant n we get:

$$\frac{d^2 r}{ds^2} = 0$$

Corresponding to propagation in straight lines.

Now consider the continuity conditions between media with different refractive indices. Since $ns(r) = n \frac{dr}{ds}$ (defining $s(r)$ is the gradient of S it follows that

$$\nabla_x(ns(r)) = 0$$

or the tangential component of ns is constant across the surface.

This amounts to

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

across an interface which is Snell's law of refraction. For the case of reflection we simply get the equality of the angles since the refractive index is identical.

What also follows from this is Fermat's principle: light rays take the shortest (actually extremal) optical path available. This is since $\oint ns(r) \cdot dr = 0$, and on a ray, s is always parallel to dr . In fact, Snell's law can be easily derived from Fermat's principle.

Example:

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{(L-x)^2 + a^2}}{v_2}$$

$$\frac{dt}{dx} = \frac{x}{v_1 \sqrt{x^2 + a^2}} - \frac{L-x}{v_2 \sqrt{(L-x)^2 + a^2}} = 0$$

$$\frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2} = 0$$

or

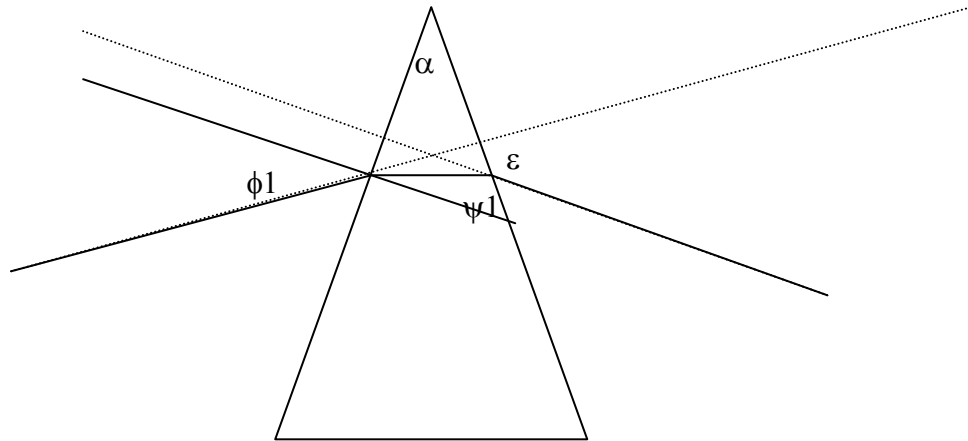
$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Note that:

1. All this is scalar (no polarization) – light was considered to be a longitudinal wave like sound waves at the time this was developed.
2. This predicts the phenomenon of total internal reflection (going from a high refractive index medium to a lower refractive index one). For example, this is widely used for high energy broadband mirrors (with right angle prisms).

2. Refraction – the prism, the thin lens, perfect imaging and projective transformations, simple optical instruments.

The simplest optical instrument which can be derived from Snell's law is the prism.



The deviation angle of a prism satisfies:

$$\varepsilon = \phi_1 + \phi_2 - \alpha$$

$$\psi_1 + \psi_2 = \alpha$$

where

$$\sin \phi_1 = n \sin \psi_1$$

$$\sin \phi_2 = n \sin \psi_2$$

Lets consider a prism-based optical instrument which deflects all rays emitted from a point into a different point – both along the same line. (For simplicity consider small angles).

Then:

$$\phi_2 = n(\alpha - \psi_1) = n\alpha - \phi_1.$$

Lets assume for simplicity a plano-convex prism. Since $\phi_1 \propto h$ and we need that $\phi_2 \propto h$, this requires $\alpha \propto h$. Since $\alpha \propto \frac{1}{h'}$, we get:

$$\frac{1}{h'} \propto h \Rightarrow h \propto \sqrt{x}$$

Which is just the spherical thin lens. Clearly, this works just as well when adding the same radius of curvature to both sides.

A simple calculation yields for the thin lens that rays from infinity are focused at a distance f such that:

$$\frac{1}{f} = (n-1) \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

Where r_i are the radii of curvature of both interfaces.

Now let's turn to a somewhat more detailed description of imaging by a lens. The lens in the above description (at least for small angles) projected a point to a point, but how does it project more complex objects (for example other points, or a line)?

A perfect imaging system is one in which every curve is geometrically similar to its image. In particular, lines are imaged onto lines. Thus, a perfect imaging system is a projective transformation. For (x, y, z) in the object plane and (x', y', z') in the image plane we should get:

$$x' = \frac{F_1}{F_0} x; y' = \frac{F_2}{F_0} y; z' = \frac{F_3}{F_0} z$$

$$F_i = a_i x + b_i y + c_i z + d_i$$

We can also write an inverse transform for x, y, z using the primes. Clearly points lying in the plane $F_0=0$ will be imaged onto infinity, and an object from infinity will be imaged onto $F'_0=0$. These are called the focal planes of the imaging system.

For the special case of an axially symmetric system we can assume $x=x'=0$, as well as symmetry for $y \rightarrow -y, y' \rightarrow -y'$. The equations then reduce to:

$$y' = \frac{b_2}{c_0 z + d_0} y \quad ; \quad z' = \frac{c_3 z + d_3}{c_0 z + d_0}$$

which contains 4 independent variables (up to ratios). The focal planes are given by: $z = -d_0/c_0$; $z' = c_3/c_0$, that is planes perpendicular to the optic axis.

Measuring distances from the respective focal planes:

$$Y = y \quad ; \quad Y' = y'$$

$$Z = z + \frac{d_0}{c_0} \quad ; \quad Z' = z' - \frac{c_3}{c_0}$$

this transforms to:

$$Y' = \frac{b_2}{c_0} \frac{Y}{Z} \quad ; \quad Z' = \frac{c_0 d_3 - c_3 d_0}{c_0^2 Z}$$

Setting:

$$f = \frac{b_2}{c_0} \quad ; \quad f' = \frac{c_0 d_3 - c_3 d_0}{b_2 c_0}$$

we easily get:

$$\frac{Y'}{Y} = \frac{f}{Z} = \frac{Z'}{f'}$$

which gives the magnification and newton's equation: $ZZ' = ff'$

One important implication of the concept of perfect imaging as a projective transformation is that an optical system containing more than one surface of revolution can be mathematically modeled as cascaded projective transformations, which is by itself a projective transformation. As such, it is clear that a set of spherical optical elements (in the paraxial limit) can be modeled by a single effective element. The simplest example is, indeed, the thin lens, which is composed of two back-to-back spherical elements, and for which we get the previous result. However, in the paraxial limit, a thick lens or a set of thin lenses can also be mathematically formulated as a projective transformation. The simplest example here is of two thin lenses with a spacing l between them, for which we can derive:

$$\frac{1}{f} = \frac{1}{f_0} + \frac{1}{f_1} - \frac{l}{f_0 f_1}$$

For $l=f_0+f_1$ this gives $f \rightarrow$ infinity, as expected. This is, in fact a telescope. For $l=0$ we get the additivity of lenses. We will now present a simple way of calculating (or proving) this formula using raytracing.

3. Ray-tracing

Imaging systems are generally, of course, not as perfect as described by a projective transformation. For a full analysis of more complicated optical systems we have to take into account higher order corrections beyond the projective transformation. In the following I start by describing one of the commonly used numerical methods (in a simplified form) to evaluate optical aberrations, ray-tracing, and follow by a formal treatment of the types of aberrations in optical systems.

Ray tracing basically follows the solution of the eikonal equation for "pencils" of rays emanating from one point to different directions. I present here a simplified form of it which is applicable to paraxial meridional rays (those which are in a plane with the optic axis) called "ABCD" matrix.

For these rays, the output ray (characterized by its distance from the optic axis and its angle) can be described by a simple 2x2 matrix as a function of the input. Deviations from paraxiality can be described numerically by introducing a nonlinear dependence into the 2x2 matrices.

In the simplest form, we can write:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

For any optical system.

Some elementary matrices are:

Propagation through free space of index n:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

Refraction through a planar interface between materials with n1 and n2:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

and refraction through a thin lens in a medium of index n:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -n/f & 1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

We can use these to derive one of the most important properties of paraxial imaging:

Since for propagation in free space and imaging all matrices have a determinant of unity, any imaging setup must also have a determinant of unity. When we have imaging, the y output is dependent only on y at the input and not on the angle. In this case, the most general ABCD matrix is:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} M & 0 \\ C & 1/M \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

The generalization of this is known as the sine condition (or the Smith-Helmholtz invariant):

$$n y \sin \theta = const$$

Let us now consider a few examples.

Starting with finding the imaging conditions for a lens of focal length f and an object placed at 2f. Up to the lens we get the matrix:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix} = \begin{pmatrix} 1 & 2f \\ -1/f & -1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

To get imaging we then require free space propagation of $d=2f$.
The final result is then:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2f \\ -1/f & -1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1/f & -1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

Corresponding to inversion with $M=1$.

Let's try now with the object positioned at $3f$. Up to the lens we get the matrix:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & 3f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix} = \begin{pmatrix} 1 & 3f \\ -1/f & -2 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

To get imaging we then require free space propagation of $d=3/2 f$.
The final result is then:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 3/2f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2f \\ -1/f & -2 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ -1/f & -2 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

Corresponding to inversion with $M=1/2$.

Let's consider now a set of two lenses spaced by a total distance l as described above.

Using the ABCD matrix formalism, the equivalent system, upon propagation from the the back aperture of the first lens to the front aperture of the second lens is just the multiplication of three matrices: $l_2 * d * l_1$

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix} = \begin{pmatrix} 1-l/f_1 & l \\ -1/f_1 - 1/f_2 + l/(f_1 f_2) & 1-l/f_2 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

The focal length is in practice defined by $\theta_{out}(y_{in})$ at $\theta_{in}=0$, from which we get:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{l}{f_1 f_2}$$

For the case of a telescope ($l=f_1+f_2$) we get:

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} -f_2/f_1 & f_1+f_2 \\ 0 & -f_1/f_2 \end{pmatrix} \begin{pmatrix} y_{in} \\ \theta_{in} \end{pmatrix}$$

corresponding to inversion and magnification.

4. Aberrations

Seidel aberrations:

To lowest order we can evaluate aberrations as small deviations from a projective transformation. Lets consider an imaging system which projects a point (x_0, y_0) to a point (x', y') , where the Gaussian image (the projective transformation) of the point is (x^*, y^*) . The ray aberration can be defined as the distance between (x', y') and (x^*, y^*) .

Let us define a useful quantity, the wavefront aberration D , from which the image aberration can be derived – as the pathlength difference between the actual wavefront and a spherical wavefront converging at the Gaussian image as a function of the exit pupil plane position (x, y) . It can be easily shown that the image aberration is just the derivative of the wavefront aberration (a simple example is a spherical wavefront impinging on a nearby point, where the pathlength difference is just a linear function).

In a cylindrically symmetric system, D is only a function of r at the object plane, ρ at the exit pupil plane, and the angle (or the scalar product κ) of the two. Since the lowest order terms can be 4th order and cannot contain an explicit dependence on r only (this violates the assumption that the image is indeed a Gaussian image), the general form of this is:

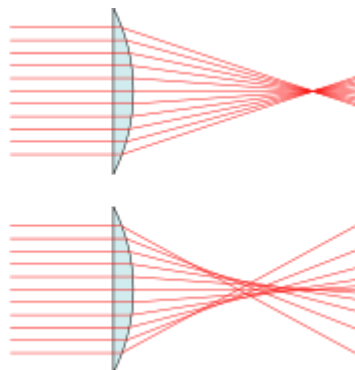
$$D = -\frac{1}{4}B\rho^4 - C\kappa^4 + \frac{1}{2}Dr_0^2\rho^2 + Er_0^2\kappa^2 + F\rho^2\kappa^2$$

The most general form of the image aberration can thus be expressed as (without loss of generality assume $x'=0$ and replace y' with y_0), simply taking the derivative of the previous expression:

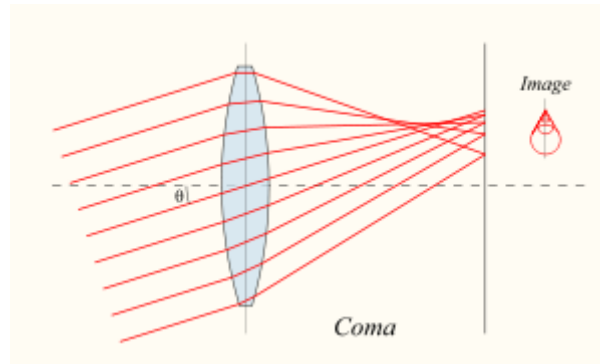
$$\begin{aligned}\Delta x &= B\rho^3 \sin \theta - 2Fy_0\rho^2 \sin \theta \cos \theta + Dy_0^2\rho \sin \theta \\ \Delta y &= B\rho^3 \cos \theta - 2Fy_0\rho^2(1 + 2\cos^2 \theta) + (2C + D)y_0^2\rho \cos \theta - Ey_0^3\end{aligned}$$

These coefficients relate to the following:

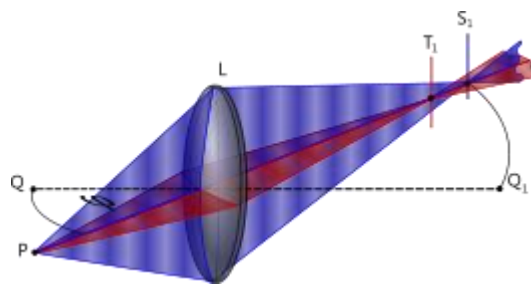
B is spherical aberration – occurs also on axis – focal length is dependent on ray direction



F is coma – smearing according to angle of incidence

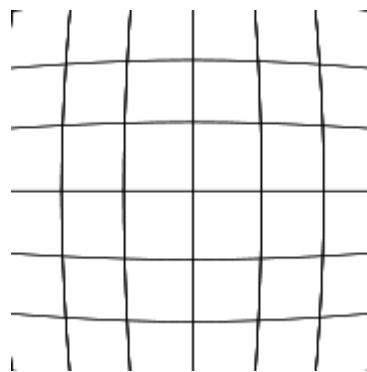


D is astigmatism – different focal length for tangential and saggital rays



C is Field curvature – the image lies on a spherical front

E is distortion – magnification is dependent on position



The elimination of aberrations is a science by itself. Let us just give a very simple example – considering the scaling of the longitudinal spherical aberration of a thin lens, simply to envision the complexity.

Let us consider a thin lens with radii of curvature r_1 and r_2 , and an object positioned at a distance S from the lens, whose image is at S' . The longitudinal spherical aberration in this case is defined as:

$$L_s = 1/S'_h - 1/S'_p$$

where p is the paraxial value and h the value for a ray at height h in the lens.

Expanding to third order for the angle of incidence on the lens we can write:

$$\gamma = \alpha + \beta$$

with

$$\tan \alpha = \frac{h}{S}; \tan \beta = \frac{h}{r_1}$$

or to third order

$$\alpha + \frac{\alpha^3}{3} = \frac{h}{S}; \beta + \frac{\beta^3}{3} = \frac{h}{r_1}$$

The deviation angle from the paraxial value

$$\sin \gamma - \gamma = -\frac{\gamma^3}{6} = -\frac{(\alpha + \beta)^3}{6} \approx -h^3 \left(\frac{1}{S^3} + \frac{1}{r_1^3} \right)$$

A similar expression can be written for S' and r2.

Expanding for a total angular deviation δ :

$$\tan \alpha' = \frac{h}{S'}; \tan(\alpha' + \delta) = \frac{h}{S' + \Delta S'}$$

Since:

$$\tan(\alpha' + \delta) \approx \tan \alpha' + (1 + \tan^2 \alpha') \delta \approx \tan \alpha' + \delta$$

We get (the last term neglecting higher order terms in h):

$$h \left(\frac{1}{S'} - \frac{1}{S' + \Delta S'} \right) = h L_s \propto h^3$$

The full expansion of this expression (containing r1, r2, S, and S', all related through f) can be simplified to the form:

$$L_s = \frac{h^2}{8f^3 n(n-1)} \left[\frac{n+2}{n-1} q^2 + 4(n+1)pq + (3n+2)(n-1)p^2 + \frac{n^3}{n-1} \right]$$

$$p = \frac{S' - S}{S' + S}; q = \frac{r_2 + r_1}{r_2 - r_1}$$

which for a given p can be minimized choosing r1 and r2 such that:

$$q = -\frac{2(n^2 - 1)}{n + 2} p$$

A condition called "best form" lens.

Origin the frequency dependent refractive index (Lorentz-Lorenz formula)

For an electromagnetic wave in a medium we can treat the electron motion in a material as a damped harmonic oscillator, and the electromagnetic field as the driving force.

Thus (neglecting the magnetic force on the electron, assuming its velocity is small):

$$m\ddot{r} + qr = eE'$$

which, for a harmonic driving field gives the solution:

$$r = \frac{eE'}{m(\omega_0^2 - \omega^2)}; \omega = \sqrt{\frac{q}{m}}$$

The contribution of the electrons to the polarization in the medium is much larger than that of the heavier nuclei, so they dominate, to give approximately:

$$P = Np = Ner = N \frac{e^2}{m} \frac{E'}{(\omega_0^2 - \omega^2)}$$

Since from Maxwell's equations $P = N\alpha E'$, we can identify α , and by a simple transformation also the polarizability η (which is the coefficient of proportionality with the external field):

$$\eta = \frac{N\alpha}{1 - \frac{4\pi}{3} N\alpha}$$

and $\varepsilon = 1 + 4\pi\eta$:

$$\varepsilon = \frac{1 + \frac{8\pi}{3} N\alpha}{1 - \frac{4\pi}{3} N\alpha}$$

Or reversing this relation:

$$\alpha = \frac{3}{4\pi N} \frac{n^2 - 1}{n^2 + 2}$$

Which for $n \sim 1$ can be approximated as:

$$n^2 - 1 \approx \frac{4\pi N e^2}{m(\omega_0^2 - \omega^2)}$$

Since for most optical materials electronic resonances occur only in the UV and MIR vibrational resonances are strongly detuned from the visible region, this means that the index of refraction in the visible range is a monotonically decreasing function of the wavelength, usually given by a phenomenological equation of the above form, termed Sellmeier equation.

Chromatic aberrations

Another type of aberration is chromatic aberration: Due to the variation of the refractive index with color, different colors have different focal lengths. This is corrected for using Doublet achromats of two glasses with different dispersion:

Since

$$\delta f = \frac{f \delta n}{n - 1}$$

For a combination of two cemented lenses we can have a solution for the two equations:

$$\frac{f_1 \delta n_1}{n_1 - 1} + \frac{f_2 \delta n_2}{n_2 - 1} = 0$$
$$\frac{1}{f_1} + \frac{1}{f_2} = F$$

In fact, there are three radii of curvature and only two equations, so that the extra degree of freedom can be used to reduce spherical aberrations.

This idea can be extended to more components such that additional derivatives of the focal length are zero, resulting in apochromats (2nd derivative also zero) and suprachromats (3rd derivative).