Lessons from Hydrodynamic Turbulence

Turbulent flows, with their irregular behavior, confound any simple attempts to understand them. But physicists have succeeded in identifying some universal properties of turbulence and relating them to broken symmetries.

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It is trite to regard turbulence as the last unsolved problem in classical physics and to cite many books and authorities to justify the opinion. It is likewise a cliché to list great physicists and mathematicians, such as Werner Heisenberg, Richard Feynman, and Andrei Kolmogorov, who "failed" to solve the problem despite much effort. Horace Lamb and others have been credited with wishing to seek heavenly wisdom on the subject when they arrived in heaven. With such lists and stories, youngsters are cautioned, directly and indirectly, that turbulence is beyond reasonable grasp.

One need not apologize for or despair over the difficulty of turbulence. For one thing, turbulence has contributed several ideas and tools of lasting value to neighboring areas of physics. A sampling includes negative temperature, anomalous diffusion, and the concept of power-law scaling in many-body problems. The powerful notions of scale invariance and universality were first proposed in the context of turbulence. In general, turbulence is a playground for solutions that are non-unique or that depend sensitively on initial conditions, and in particular the subject provided the context in which the problem of predictability was first posed in concrete terms. Turbulent flows allowed physicists to recognize and unambiguously express both the coexistence of structure and randomness and the role of correlated structures in the transport of matter, heat, and momentum.

Underlying the above comments is the notion that turbulence is not a single problem but rather a huge field with pivotal applications in engineering, geophysics, astrophysics, and cosmology. It is also an excellent source of problems for pure mathematics. Applied mathematicians, meteorologists, and engineers often focus on particulars like drag and pressure drop, mean velocity distributions, mixing efficiencies, and dispersion rates. Indeed, by considering the totality as "the problem of turbulence," one can justify the claim that the problem remains unsolved. In like manner, the insistence on a complete first-principles understanding of the structure of complex atoms and molecules might lead to the conclusion that quantum mechanics is an unsolved problem.

The diversity of problems in turbulence and their specific complexities should not obscure the fact that the heart of the subject belongs to physics. The central difficulties in the field are those of strong fluctuations and strong coupling—field theory and condensed matter physics confront the same problems. Scientists have made substantial progress in addressing those difficulties, and now understand a few fundamental properties of turbulence. In this article we present in particular the concept of statistical conservation laws and describe their role in the breaking of turbulent-state symmetries. Those powerful laws enable one to elucidate the concept of universality in turbulence and also, perhaps ironically, highlight the limitations of the concept.

The inertial range

Although solids and plasmas can display turbulence, we will limit our attention to fluids. One mechanism for the generation of turbulence is shown in figure 1. Turbulence is unrepeatable in detail and is irregular in both time and space. It can be sustained against dissipation only by an external force or by the addition of energy at the boundary, without which it will eventually decay. Turbulent flows have many strongly interacting degrees of freedom and are far from equilibrium.

Figure 2 illustrates a fluid confined in a tank and set in motion by an external stirrer whose diameter is comparable in size to that of the tank. One might expect the resulting motion to depend on the fluid density and its viscosity ν, and on the characteristic speed V and diameter L of the stirrer. However, the fluid density is essentially constant within the flow, and all properties, including the power supplied by the stirring blade, can be normalized by that density. With that normalization convention, we eliminate density from consideration. The nature of the blade and tank boundaries could play some role, but we focus on the state of the flow far from boundaries, where we assume that blade details and boundary effects are negligible. Given that assumption, the only parameters that affect the flow are L, V, and ν, from which one can construct the dimensionless Reynolds number $Re \equiv LV/\nu$.

Viscosity is an internal friction; thus when Re is small, viscosity damps out variations in the velocity, and the flow is smooth. On the other hand, when Re is large (how large depends on the details of the flow configuration), the motion set up by the blade is dominated by nonlinear effects. One of those effects is the production of smaller-scale motions, which in turn produce motions on even smaller scales. Eventually, a hierarchy of scales appears in the flow. Turbulence in the steady state is the collection of those many-scaled motions.

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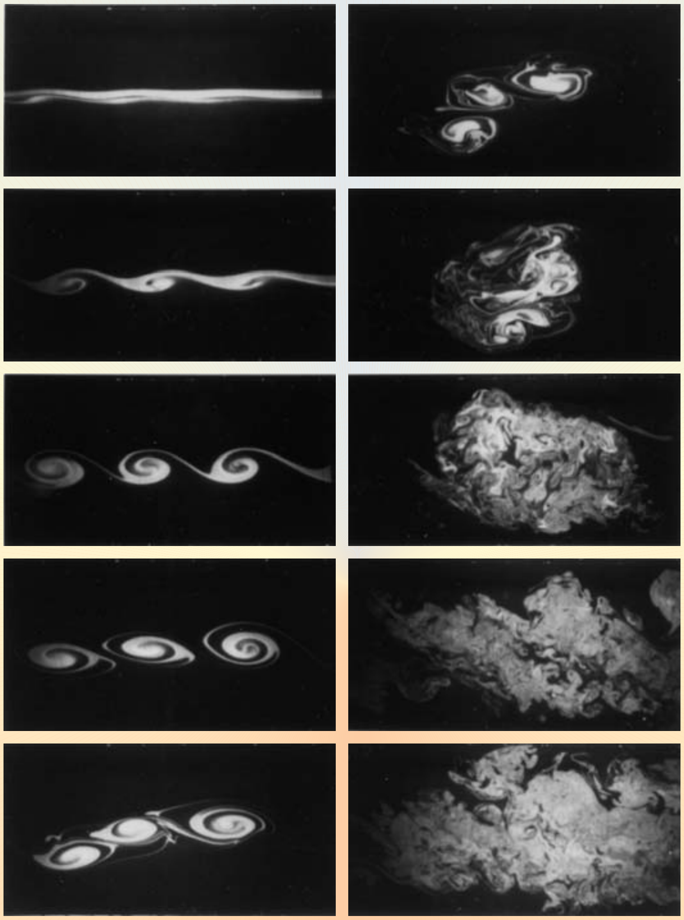

Figure 1. The transition to turbulence is visible in this series of photographs for which time increases from top to bottom, left to right. This particular process shows the initiation of the Kelvin–Helmholtz instability in a shear flow, which is by no means the only way to generate turbulence. The bright region marked by dye is sheared by opposing uniform streams of fluid, which appear dark in the photos. The dyed region becomes increasingly wavy, then, due to nonlinearities, rolls up into distinct vortices that merge with neighbors and eventually become turbulent. Although we understand well the first few steps in the evolution, we still only poorly understand how the turbulence is generated.

The generation of smaller scales occurs increasingly rapidly as the scale size decreases, which is why the process is figuratively described as a cascade. The cascade is terminated by viscosity at a characteristic length scale η, at which gradients in the fields that describe the fluid flow are the sharpest. One may increase the gap between L and η by increasing Re at fixed L. At intermediate scales distant from both L and η—the so-called inertial range—one might expect the properties of turbulence to be universal, that is, indifferent both to how energy is injected at the scale L and to how it is dissipated at η. An idea put forth by Lewis Richardson about 80 years ago is consistent with that indifference postulate; in his formulation,[1] the kinetic energy injected at L is cascaded without loss through the inertial range and is dissipated by viscous action at scales of order η. Indeed, that nonlinear energy transfer process is a dominant feature in the inertial range. For additional details, see the box on page 45.

The conceptual simplicity associated with the inertial range makes it natural to ask if the notion of universality in that range can be elevated to a physical law. If so, different turbulent flows would share common features. The quest for universality is motivated by a hope of identifying general rules that govern far-from-equilibrium systems, principles similar in scope to the variational principles that govern thermal equilibrium. Since all dynamical features of turbulence are irregular, questions about turbulent flows can be posed and answered only in terms of statistical averages.

Scale invariance and universality

Those who study turbulence believe that all its important properties are contained in the Navier–Stokes equations for fluid motion. For a unit-density fluid, the equations assume the form

$$\mathbf{v}_t + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + \mathbf{f} + \nu\Delta\mathbf{v}.$$

Here $\mathbf{v}$ is the velocity vector, p is the local pressure, $\mathbf{f}$ is a random forcing that abstracts the effect of boundary conditions on obstacles or other ways of generating turbulence, and the suffix t stands for partial time derivative. The Navier–Stokes equations are supplemented by the incompressibility condition $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$.

To study motion in the inertial range, it is convenient to consider the structure functions—moments of velocity differences δv_r across an inertial-range scale of size r. Structure functions of order n are defined by $S_n \equiv \langle\{[\mathbf{v}(\mathbf{r},t) - \mathbf{v}(\mathbf{0},t)]\cdot\mathbf{r}/r\}^n\rangle \equiv \langle(\delta v_r)^n\rangle$, with r the magnitude of the distance $\mathbf{r}$ between two points and angle brackets denoting a suitable average. The second-order structure function, for example, is the energy contained in all Fourier modes with wavenumbers larger than $1/r$.

The third-order structure function is of special interest because, as Kolmogorov showed[2] starting with the Navier–Stokes equations, a flux-constancy condition holds exactly for r in the inertial range. His result, $S_3/r = -\tfrac{4}{5}\langle\epsilon\rangle$, is known as the four-fifths law. The quantity $\langle\epsilon\rangle$ is the average rate of energy dissipation per unit mass and can be identified with the energy flux across scales. Note that the definition of S_3 includes three powers of velocity, a quantity that changes sign under the operation of time reversal. Thus, the nonvanishing of S_3 signals a breakdown of time reversibility in the inertial range.

No one has yet been able to deduce closed-form expressions for moments of other orders. Nonetheless, Kolmogorov attempted a giant step and inferred that the structure functions S_n are proportional to r raised to the power $\zeta_n = n/3$, linear in the moment order.[2] That result, consistent with the four-fifths law, exemplifies the principle of scale invariance in three-dimensional turbulence.

If Kolmogorov's scale invariance were exact, the task of computing practical flows would be relatively simple. Airflows over an aircraft fuselage or within a cloud, for example, have about 10^{18} excited degrees of freedom, some of which must be modeled statistically if the flow is to be computable. Scale invariance would have provided a convenient framework for developing such models.

Anomalous scaling

Modern evidence shows that the scaling exponents of structure functions depart from Kolmogorov's aesthetically appealing result. The relative difference between the measured exponents and Kolmogorov's prediction is shown in figure 3, which also displays the results of several recent calculations and simulations.[3] The breakdown of scale invariance in the inertial range, now called anomalous or multifractal scaling, is an important feature of turbulence. Unlike for critical phenomena, one needs to work out the behavior of turbulence structure functions of each order independently, without recourse to the dimensional analysis that yields $\zeta_{2n} = n\zeta_2$.

In turbulence, $\zeta_{2n} < n\zeta_2$, so that for $n \geq 2$, $S_{2n}/(S_2)^n$ increases as $r \to 0$. The relative growth of high moments means that strong fluctuations become more probable as the scales become smaller. The practical importance of high-moment growth is that it limits one's ability to produce realistic models for small-scale turbulence.

If scale invariance cannot be used to compute structure function exponents of all orders, what *is* possible? Conservation laws impose constraints on the dynamics, and so conserved quantities or their fluxes play an essential role in answering the question. In fluid turbulence, energy conservation is broken at the large scale through the addition of energy by stirring, for example, and at the scales of sharp gradients through dissipation by viscosity. Neither energy input nor dissipation dominates the inertial range, and energy is conserved there. However, unlike in equilibrium systems in which energy is equally partitioned over all scales, energy conservation in the inertial range of turbulence means the energy flux across all scales is constant.

It might appear from the Navier–Stokes equations that the dissipation rate of turbulent energy, $\langle\epsilon\rangle$, would vanish as the viscosity approaches zero. But an important feature of turbulence is that $\langle\epsilon\rangle$ remains finite in the limit of vanishing viscosity. No matter how small the viscosity, how high the Reynolds number, or how extensive the range of scales participating in the energy cascade, the energy flux remains equal to that injected at the stirring scale. That surprising result is probably the first example of what is called an anomaly in modern field-theoretical language: An inviscid-equation symmetry—in this case, time-reversal invariance—remains broken even as the symmetry-breaking viscosity becomes vanishingly small.[4] A trained eye viewing a movie of steady turbulence run backwards can tell that something is indeed wrong!

Recall that the third-order structure function is determined completely by the energy flux, and that the flux's nonzero value signals the breakdown of time-reversal symmetry in the inertial range. Thus one might ask, Are there other conserved-quantity candidates whose broken symmetries yield structure functions of other orders? That question is fundamental for turbulence, indeed for modern statistical physics, and provides the thread that we follow in this article. We first consider a relatively simple 1D model.

Burgers turbulence

John von Neumann recognized that some important general features of turbulence can be understood by studying the equation

$$u_t + uu_x = \nu u_{xx},$$

first suggested by Johannes Burgers.[5] The Burgers equation, though 1D, retains some important properties of the Navier–Stokes equations. It describes weakly compressible 1D flows as well as stochastic surface growth and other systems, and has been studied in much detail lately.[5]

For zero viscosity, the Burgers equation conserves an infinity of dynamical integrals of motion, $E_n = \frac{1}{2}\int u^{2n}\,dx$. For small viscosity, the equation has a propagating shock wave as its solution. If one were to add a random forcing term that is correlated on large scales, then nonlinearities

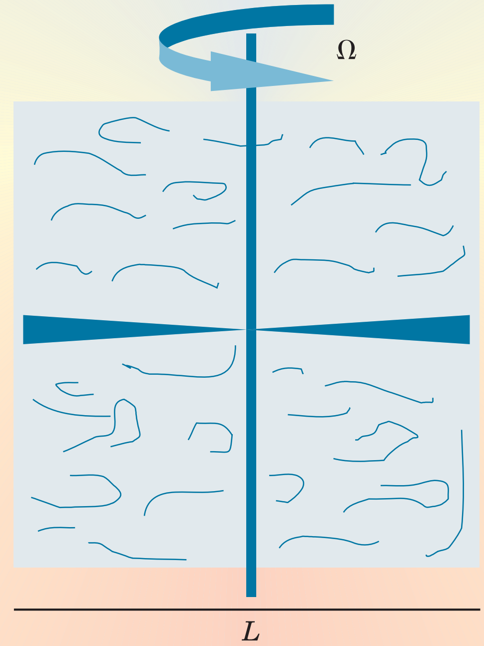

Figure 2. Fluid in a tank of diameter L is stirred by a blade of similar size. The stirrer rotates with an angular frequency Ω and initiates motion at the large scale L with characteristic speed $V \propto \Omega L/2$. In time, nonlinear interactions generate motions at increasingly smaller scales until, at the smallest scales, the energy introduced by the stirrer is dissipated.

The Energy Cascade

A liquid confined to a tank of diameter L and stirred by a propeller of comparable diameter can be used to illustrate the notion of an energy cascade. The propeller adds energy to the system, mostly at the large scale L, and that energy dissipates at the small scale η determined by the size and speed of the propeller and the viscosity of the fluid. This simple description leads one to expect an energy flux from the large to the small scale. In 1941, Andrei Kolmogorov derived a quantitative law—the four-fifths law discussed in the text—that is interpreted in terms of energy flux; he showed that the flux averaged over time is constant across all scales between L and η. Of course, the energy cascade is unidirectional only on the average: At some instants of time, energy can pass from small to large scales.

With a certain degree of crudeness, one can imagine the energy cascade as being analogous to the flow of water through a pipe whose radius decreases continuously from L to η. Since matter is neither destroyed nor produced inside the pipe, constant flux requires that the speed of the water increase as the fluid moves toward the small scales. Similarly, energy transfer at the small scale occurs faster than at the large scale.

Water, of course, won't flow through a sealed pipe. But as long as η is not zero, water must exit the pipe's small section at the same rate as it enters the largest section. That observation is analogous to the dissipative anomaly in turbulent systems: No matter how small the viscosity that determines the small scale η, the energy dissipation rate at η must equal the introduction rate at L. However attractive the notion of energy cascades, though, it must be taken mostly as pedagogical imagery.

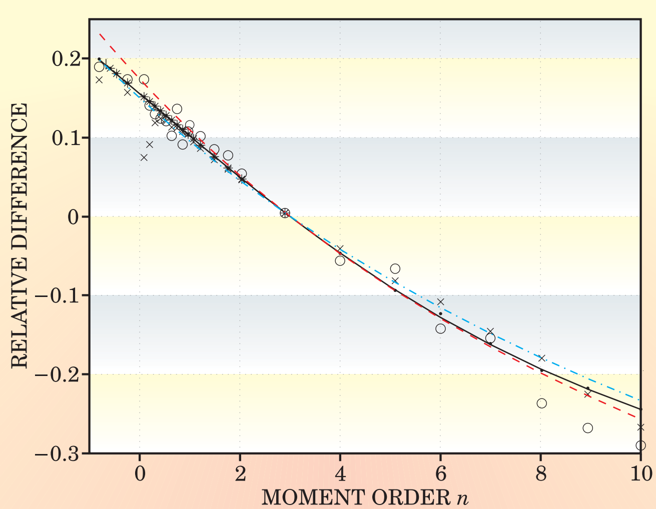

Figure 3. Scaling exponents ζ_n differ from the scale-invariant, linear Kolmogorov value of $n/3$. In this plot, the relative difference is defined as $(\zeta_n - n/3)/(n/3)$, open circles show experimental results, crosses and stars show results of simulations, and the curves give various theoretical results. Note that the relative difference passes through $n = 0$ without showing any special feature. Nor does there appear to be any special behavior as n approaches −1, notwithstanding that structure functions of order −1 and lower are undefined. (Adapted from ref. 3.)

would cause the spatially smooth acoustic waves initially produced to evolve into shocks.

The shocks provide the mechanism for dissipation rates $\langle\epsilon_n\rangle$ appropriate to the integrals E_n. One property of shock dissipation is that the $\langle\epsilon_n\rangle$ of various orders tend to nonzero values as the viscosity tends to zero, and it is easy to show that $S_{2n+1}(x) = -4(2n+1)\langle\epsilon_n\rangle x/(2n-1)$ for all n. The simple scaling, $S_{2n+1}(x) \propto x$, arises because the probability of having a shock within an interval of size x is proportional to x but the velocity difference across a shock is independent of x.

In analogy to the 3D Navier–Stokes case, $S_3(x) = -12\langle\epsilon_1\rangle x$. The structure function S_3 exhibits a universal behavior that is determined solely by $\langle\epsilon_1\rangle$ and, in particular, that is independent of the nature of the forcing. Scale invariance suggests that other structure functions $S_n(x)$ would be given by $(\langle\epsilon_1\rangle x)^{n/3}$, but that is not the case. The failure to obtain scale-invariant structure functions means that small scales, however small, "remember" an infinity of input rates $\langle\epsilon_n\rangle$ determined by the forcing. The bottom line: The breakdown of scale invariance in Burgers turbulence is related to an infinity of inviscid constants of motion and to the shocks that are responsible for symmetry breaking by dissipation.

Statistical conservation laws

We now describe conservation laws that are qualitatively different. They are conserved only on the average, yet determine the statistical properties of strongly fluctuating systems. In random systems, it is always possible to find fluctuating quantities with invariant averages, but our question is more subtle: Is it possible to find quantities that may be expected to change on dimensional grounds but nonetheless stay constant?

One can describe n fluid particles in a random flow in terms of interparticle distances so that R_{ij} denotes the distance between particles i and j. Consider functions of interparticle distance that satisfy $f(\lambda R_{ij}) = \lambda^{\zeta} f(R_{ij})$. Such functions are called homogeneous functions of degree ζ. When distances grow on the average according to $\langle (R_{ij})^2\rangle \propto t^a$, say, one would expect on dimensional grounds that a generic homogeneous function f would grow as $f \propto t^{a\zeta/2}$. On the other hand, as particles move in a random flow, fluctuations in the shape of the cloud of particles could decrease in magnitude. Therefore, one may look for suitable functions of size and shape that have the property of being conserved on average because the decrease of shape fluctuations compensates for the growth in size. Figure 4 illustrates the notion of size and shape compensation.

For the simple case of Brownian diffusion, the time derivative of the mean of any function of distances between particles is the Laplacian of that function. By definition, harmonic functions, or zero modes, are solutions obtained by setting the Laplacian to zero and so are statistically conserved. They are polynomials in $(R_{ij})^2$ or t, and because particles undergoing Brownian diffusion move independently, the degree $\zeta_n = 2n$ of the n-particle mode depends linearly on n. Thus, laws governing the decrease of shape fluctuations determine the exponents of zero modes. It is no coincidence that we have introduced the same notation for the degrees of zero modes and the scaling exponents discussed earlier; in the next section we will explicitly describe the connection between the two ideas.

Zero modes exist for turbulent diffusion and for Brownian diffusion, but the two types of motion have a major difference: The velocities of different particles in turbulence are correlated. Those mutual correlations make shape fluctuations decay more slowly than t^{-n} so that the exponents ζ_n of the zero modes grow more slowly with n than do the linear exponents of Brownian motion. Indeed, power-law correlations of the velocity field lead to superdiffusive behavior of interparticle separations. In other words, the farther apart the particles, the faster they tend to move away from each other. The system behaves as if particles were subject to an attraction that weakens with distance. But there is no physical interaction among particles, only mutual correlations. Zero modes of multiparticle evolution exist for all velocity fields, from smooth fields to extremely rough ones as in Brownian motion. But anomalous scaling arises only from nonsmooth velocity fields with power-law correlations in space.

Three different groups more or less simultaneously discovered the importance of zero modes[6] for the so-called Kraichnan model.[7] In essence, they found the statistically

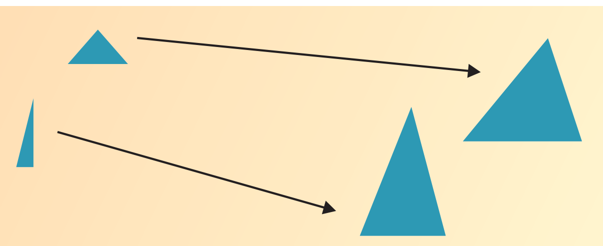

Figure 4. Size and shape fluctuations can tend in opposite directions. Illustrated here are two configurations, each with three particles at the vertices of triangles; the particles move in a turbulent flow from left to right. The sizes of the triangles increase with time but the fluctuations in their shape decrease; that is, they become more like equilateral triangles.

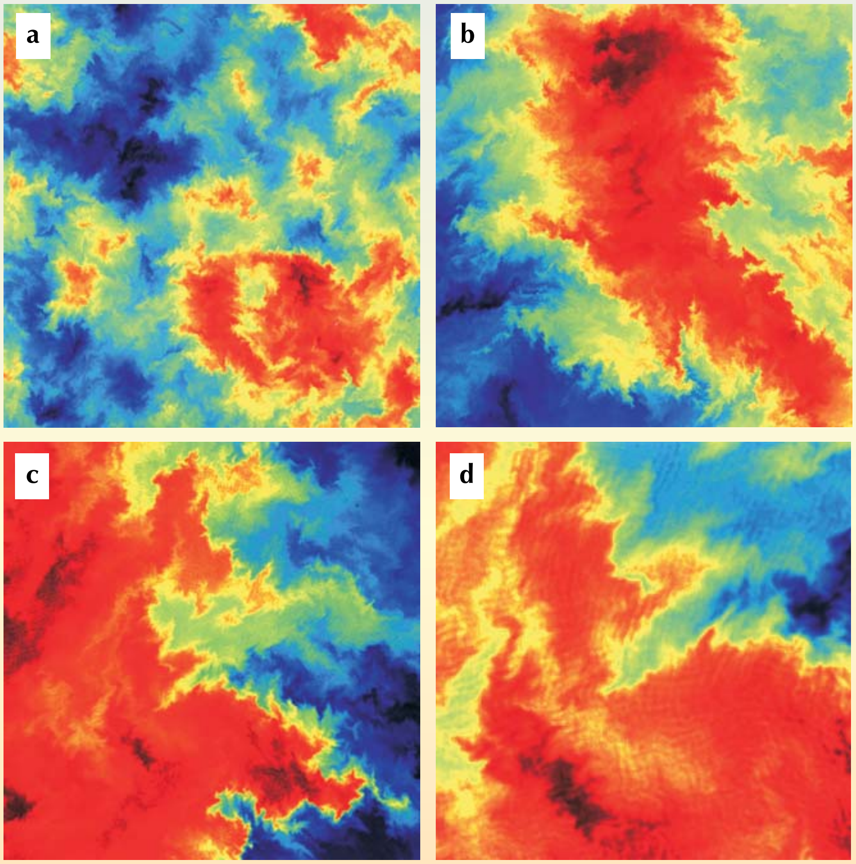

Figure 5. Scale invariance is manifested in this sequence of snapshots of a two-dimensional passive scalar field evolved in a particular type of Kraichnan model. The colors indicate the value of the field, red being high and blue low. The four panels in the sequence **a–d** show increasingly magnified views of a fixed location at a specific time. Clearly the four images are not identical. But they are similar in a statistical sense, and that notion of statistical self-similarity can be made quantitative.[18] (Illustrations generated by Inigo San Gil from a program developed by Shiyi Chen.)

conserved quantities with an anomalous scaling by determining the zero modes of the differential operator of the corresponding turbulent diffusion.[8]

Anomalous scaling of scalar fields

How do statistical conservation laws lead to anomalous scaling of fields advected by turbulence? Consider first a passive scalar field $\theta(\mathbf{r}, t)$, such as the temperature in a mildly heated flow; "passive" means that the field is carried by the flow $\mathbf{v}(\mathbf{r}, t)$ but does not affect it. In the presence of diffusive action due to κ, and sustained by an external forcing $\phi(\mathbf{r}, t)$, the field obeys

$$\theta_t + (\mathbf{v} \cdot \boldsymbol{\nabla})\theta = \phi + \kappa \Delta \theta.$$

The analogue of the large scale L of a fluid stirred in a tank is the correlation scale L_θ of the forcing, and the analogue of the dissipation scale set by viscosity is a diffusion scale η_d determined by $\delta v(\eta_d)\eta_d = \kappa$. If the correlation scale is much larger than the diffusion scale, the passive field will cascade in an intermediate range, somewhat like energy cascades in the stirred fluid: The forcing produces large-scale fluctuations of the field, which, through distortion by velocity gradients, develops increasingly smaller scales until diffusion smears them out at η_d.

For scales larger than η_d, the correlation functions of θ are proportional to the times spent by the particles within L_θ. The structure functions of θ are differences of correlation functions with different initial particle configurations as in, for instance, $S_3(r_{12}) \equiv \langle[\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)]^3\rangle = 3\langle\theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2) - \theta(\mathbf{r}_1)\theta^2(\mathbf{r}_2)\rangle$. Because fluid particles are ultimately responsible for transporting the scalar field, the structure function S_3 represents a comparison of two histories: One begins with two particles initially close to the position $\mathbf{r}_1$ and the third particle at $\mathbf{r}_2$; the other starts with one particle at $\mathbf{r}_1$ and the two other particles at $\mathbf{r}_2$. Put another way, S_3 is proportional to the time during which the two histories can be distinguished, or to the time needed for an elongated triangle to relax into the equilateral shape. That time decreases as r_{12} grows; the farther away the particles, the faster they lose correlations. The time for shape decorrelation decreases with the separation as a power law whose exponent is the same as that of the zero mode.

Strictly speaking, the flow $\mathbf{v}$ that carries the passive scalar field should be a solution of the Navier–Stokes equations. But one can work out quantitative details by assuming the flow to be a random field with an infinitesimally small correlation time and power-law correlations in space. That move is the crux of the model proposed by Robert Kraichnan, whose insight was that spatial rather than temporal correlations matter for anomalous scaling. The results we will describe, though, may well be relevant also for more general scenarios.[8,9]

In the Kraichnan model, one can analytically derive[8] the result that $S_n(r) \propto r^{\zeta_n} L_\theta^{[n(1-a)-\zeta_n]}$. Since ζ_n grows with n more slowly than linearly, $S_{2n}/(S_2)^n \propto (L_\theta/r)^{(n\zeta_2 - \zeta_{2n})}$ grows as r tends to zero; strong fluctuations of the scalar become more probable. In the unforced and undamped case, the integrals $\int \theta^n d\mathbf{r}$ are conserved, just as the integrals E_n are conserved by the inviscid and unforced Burgers equation. However, unlike in the Burgers case, the statistics of the scalar field are determined by statistical geometrical conservation laws, not by dynamical conservation laws.

Because statistical conservation laws break the scale invariance of the scalar field in the inertial range, the field is sensitive to the details of the spatial correlations in the forcing function. Furthermore, if the forcing also breaks the isotropy of the scalar field, the anomalous scaling implies that isotropy will not be restored as r tends to zero.[10] As a result, the scalar differences could get more anisotropic in the small-r limit.

The persistence of anisotropy in the limit of vanishing r is an example of an anomaly induced by a statistically conserved quantity. Other such anomalies include the broken scale invariance that remains broken as r tends to zero and the failure of time reversibility to be restored in the limit of vanishing diffusion. The anomalies associated with statistically conserved quantities are qualitatively different from those produced by dynamically conserved quantities. For example, dissipation is a singular perturbation that breaks the conservation of dynamical integrals of motion and imposes a flux-constancy condition that is similar to quantum anomalies.[4] The flux constancy, in turn, is related to cascades of conserved quantities in the inertial range. Zero modes, in contrast, have no associated cascades, nor is their conservation broken by dissipation. Anomalous scaling of zero modes is due to correlations between different fluid trajectories. As different as they are, though, the two types of anomalies are intimately related: Flux constancy imposes certain scaling properties on the velocity field that generally lead to super-diffusion and to anomalous scaling of zero modes.

If the scale L at which energy is introduced is much greater than L_θ, then the scale range $L \gg r \gg L_\theta$ corresponds to thermal equilibrium with power-law correlations—a direct analogue of critical phenomena. In that case, one finds an anomalously slow decay of correlations with the scale, as in quantum field theory and statistical physics, and can identify the statistically conserved quantities responsible for that anomalous scaling.[9] Passive scalars thus allow one to relate the breakdown of scale invariance to statistical integrals of motion both in turbulence and in equilibrium.

The emphasis on particle trajectories has brought a significant advance in numerical simulations of turbulence. By considering only a few trajectories, rather than the whole velocity field, investigators can model the phenomena that belong to the domain of high-Reynolds-number turbulence. In particular, for scalar fields, the scaling exponents approach a constant value as n gets large. That saturation can be interpreted in terms of sharp fronts.[11]

Statistical conservation laws are also associated with vector fields. In that case, the conserved quantity may involve both the coordinate of the fluid particle and the vector that it carries. In general, an increase in the distance between particles offsets a decrease in correlations between the vectors they carry. For the Kraichnan model, conservation laws have been identified for both active and passive vector fields.[8]

Two-dimensional turbulence

Large-scale motions in the atmosphere and shallow layers of fluid are nearly two-dimensional.[12] In 2D flows, the vorticity $\omega \equiv \nabla \times \mathbf{v}$ is perpendicular to the velocity so that the stretching of vortex lines by the velocity field is absent. The curl of the Navier–Stokes equations yields

$$\omega_t + (\mathbf{v} \cdot \nabla)\omega = \tilde{f} + \kappa \Delta \omega,$$

where $\tilde{f} = \nabla \times \mathbf{f}$. If the right-hand side vanishes, then the integrals $\Omega_n \equiv \int \omega^n d\mathbf{r}$ are invariant. Among them, the integrated squared vorticity, or enstrophy, $\Omega_2 = \int \omega^2 d\mathbf{r}$, like the kinetic energy E, is arguably the most basic quantity. The existence of two quadratic and positive invariants—the kinetic energy and the enstrophy—means that the steady state of turbulence must have two cascades. When one excites turbulence at a large scale by injecting energy and enstrophy at finite rates, energy does not cascade toward small scales. The reason is the exact relation $\langle \epsilon \rangle = \nu \langle \omega^2 \rangle$ in homogeneous turbulence says that nonzero energy dissipation implies infinite enstrophy dissipation in the inviscid limit. Thus, energy flows upscale in an inverse cascade while enstrophy directly cascades downscale.[13]

The previous conclusion is a nonequilibrium development of Lars Onsager's equilibrium treatment, in which joint conservation of energy and enstrophy leads to the notion of negative temperature.[14] Temperature is negative when the available phase-space volume *decreases* with increasing energy. The phenomenon arises at sufficiently high energy because the nonvanishing enstrophy requires energy to be redistributed only among modes with low wavenumbers. Kraichnan discovered the velocity spectrum in the inverse cascade for 2D incompressible turbulence and Vladimir Zakharov developed it for wave turbulence.[13] It is one of the most important results in turbulence since Kolmogorov's 1941 work.

What about the conserved quantities other than enstrophy? The intuition developed so far might suggest that the infinity of dynamical conservation laws must bring about anomalous scaling. Turbulence, though, never fails to defy expectations. Consider first the direct cascade. The constancy of the enstrophy flux takes the form $\langle(\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2)\omega_1\omega_2\rangle = \langle \varepsilon \rangle$, with subscripts referring to different locations. The above equation suggests that the velocity difference $\delta v_r \propto r$, or that the velocity field is smooth. Clearly, a completely smooth velocity field cannot generate a nonzero vorticity dissipation in the inviscid limit, but the enstrophy-flux result suggests that a logarithmic singularity in the vorticity field is enough. Indeed, particles in a smooth flow separate exponentially, so the separation time behaves as the logarithm of distance. Therefore, were ω a passive scalar, it would have logarithmic correlation functions in a smooth velocity.[15]

In fact, the vorticity is not passive but rather is related to the velocity. Still, one can treat it as a passive scalar, but the effective stretching rate acting on it must then be renormalized with the scale.[13,16] Experiments and numerics support the notions of logarithmic behavior of correlation functions and the absence of anomalous scaling in the direct cascade.

For the inverse energy cascade, the analogue of Kolmogorov's flux-constancy relation is $S_3 = \tfrac{3}{2}\langle \epsilon \rangle r$. Analysis of higher moments shows that they should be anomaly-free, and experiments indeed confirm that expectation.[12]

In sum, 2D turbulence appears to be scale-invariant at the scales both much larger and much smaller than the energy injection scale. Figure 5 illustrates a specific example.

Beyond dimensional reasoning

Kolmogorov's scale-invariant theory, like Lev Landau's theory of phase transitions, had swayed the turbulence community for many years because of its beauty and simplicity. In time, a large body of experiments suggested that Kolmogorov's pathbreaking work was inadequate; still, the various caveats associated with the experiments left room for questions.

In critical phenomena, Onsager's solution of the Ising model, though it was not particularly highly regarded at the time, played an important role in convincing physicists that Landau's mean-field theory had to be replaced. In that spirit, the turbulence community has, for many years, sought an "Ising model" of turbulence—a solvable model in which the essential complexity of the problem is maintained. The Kraichnan model and Burgers turbulence have offered some successes. One can now state with confidence that stochastic differential equations like those that describe aspects of turbulence demonstrate the inadequacy of Kolmogorov's dimensional reasoning. In particular, the community has learned that statistical conservation laws play a fundamental role in establishing that inadequacy.

Briefly, the statistically conserved quantities involve the geometry of multipoint configurations of fields advected by the flow. If anomalous scaling is to result, the advecting velocity field must not be smooth and it must generally possess power-law correlations in the inertial range. Those properties produce correlations between fluid particles that, in turn, make the scaling exponents of the statistical conservation laws nonlinear functions of the particle number.

We believe that these lessons of hydrodynamic turbulence are widely valid. That is, they hold both in and out of equilibrium for other nonlinear multidimensional systems that possess statistical conservation laws and anomalous exponents. We hope that future research will discover additional fundamental links between turbulence, critical phenomena, and other problems of condensed matter physics and field theory. In the end, as Philip Anderson wrote, "Physics [must be] well embedded in the seamless web of cross-relationships."[17]

References

1. L. F. Richardson, *Proc. R. Soc. London, Ser. A* **110**, 709 (1926).
2. A. N. Kolmogorov, *Dokl. Akad. Nauk. SSSR* **30**, 9 (1941); **32**, 16 (1941), reproduced in *Proc. R. Soc. London, Ser. A* **434**, 9 (1991).
3. S. Chen et al., *J. Fluid. Mech.* **533**, 183 (2005).
4. K. R. Sreenivasan, *Phys. Fluids* **27**, 1048 (1984). The analogy between dissipative anomalies in turbulence and the axial anomaly associated with the breakdown of chirality conservation in quantum field theory was noticed by A. M. Polyakov, *Nucl. Phys. B* **396**, 367 (1993).
5. J. von Neumann, in *Collected Works (1949–1963)*, vol. 6, A. H. Taub, ed., Pergamon Press, New York (1963), p. 437. See J. M. Burgers, *The Non-Linear Diffusion Equation: Asymptotic Solutions and Statistical Problems*, Reidel, Boston (1974); W. E. K. Khanin, A. Mazel, Ya. G. Sinai, *Phys. Rev. Lett.* **78**, 1904 (1997); *Ann. Math.* **151**, 877 (2000); U. Frisch, J. Bec, in *New Trends in Turbulence. Turbulence: nouveaux aspects, Les Houches Session 74* (2000), M. Lesieur, A. Yaglom, F. David, eds., Springer, New York (2001), p. 341.
6. B. Shraiman, E. Siggia, *C. R. Acad. Sci., Ser. II* **321**, 279 (1995); K. Gawędzki, A. Kupiainen, *Phys. Rev. Lett.* **75**, 3834 (1995); M. Chertkov, G. Falkovich, I. Kolokolov, V. Lebedev, *Phys. Rev. E* **52**, 4924 (1995).
7. R. H. Kraichnan, *Phys. Fluids* **11**, 945 (1968).
8. G. Falkovich, K. Gawędzki, M. Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001).
9. A. Celani, M. Vergassola, *Phys. Rev. Lett.* **86**, 424 (2001); G. Falkovich, A. Fouxon, *Phys. Rev. Lett.* **94**, 214502 (2005); A. Celani, A. Seminara, *Phys. Rev. Lett.* **94**, 214503 (2005).
10. K. R. Sreenivasan, *Proc. R. Soc. London, Ser. A* **434**, 165 (1991).
11. U. Frisch, A. Mazzino, M. Vergassola, *Phys. Rev. Lett.* **80**, 5532 (1998); O. Gat, I. Procaccia, R. Zeitak, *Phys. Rev. Lett.* **80**, 5536 (1998); A. Celani, A. Lanotte, A. Mazzino, M. Vergassola, *Phys. Rev. Lett.* **84**, 2385 (2000).
12. P. Tabeling, *Phys. Rep.* **362**, 1 (2002).
13. G. K. Batchelor, *The Theory of Homogeneous Turbulence*, Cambridge U. Press, New York (1982); R. H. Kraichnan, *Phys. Fluids* **10**, 1417 (1967); V. Zakharov, "Some Questions on the Nonlinear Theory of Surface Waves," PhD thesis, Novosibirsk Nuclear Physics Institute (1967).
14. L. Onsager, *Nuovo Cimento Suppl.* **6**, 279 (1949).
15. G. K. Batchelor, *Phys. Fluids Suppl. II* **12**, 233 (1969).
16. G. Falkovich, V. Lebedev, *Phys. Rev. E* **50**, 3883 (1994).
17. P. W. Anderson, *Stud. Hist. Phil. Mod. Phys.* **32**, 487 (2001).
18. S. Chen, R. H. Kraichnan, *Phys. Fluids* **10**, 2867 (1998). ■